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A New Approach for Ranking Efficient DMUs Based on Network Data Envelopment Analysis and Borda Technique

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Abstract

Classic Data Envelopment Analysis methods ignore the internal interactions and consider the systems as a 'black box'. Most of the network analysis models are nonlinear and it is feasible that this point may cause a considerable amount of modification to occur in the efficiency results. Amidst which, models, such as, the Wang Model, takes the intermediate variables into account, but in order to prevent intricacies in resolving models, it has an inconclusive approach, between two outlooks, of the black box and the network. Hence, in this paper, we consider a three-stage network with additional, desirable and undesirable inputs and outputs and the three abovementioned approaches are analyzed by contemplating on the optimistic and pessimistic views. The goals of this paper are to put together the results of the three mentioned approaches in order to attain the final conclusions. We generalize and use of Wang's approach for the three levels, with additional inputs and outputs, as well as a heuristic solution to solve the network's view. Finally, this paper considers a genuine world example, in the form of a dynamic network, for model application and analyzes it from three perspectives.

Keywords: Data envelopment analysis, Network DEA, Three-stage. Double-frontier, Undesirable outputs, Borda technique

Paper Type: Original Research

1. Introduction

DEA is a non-parametric frontier method to assess the relative efficiency of a set of comparable entities. These entities are called Decision Making Units (DMUs) with multiple inputs and outputs (Kabnurkar 2001; Saati et al. 2012; Shi et al. 2021). DEA goals are to measure efficiency and present the efficient and inefficient units. Efficient units generate the maximal number of outputs from the minimal number of inputs and their efficiency equates to (1). These units form the efficiency frontier and the other units which have an efficiency ranging from (0 to 1) are identified as inefficient units (Amirteimoori et al. 2006; Ebrahimnejad et al. 2015). The DEA has such capacities, where each DMU, can be considered in segregation and specifies the efficiency appraisal on the basis of the distance of this unit till the efficiency frontier (Kao & Hwang 2008). The DEA entitled 'Rodez's Research', which, with the collaboration of Cooper and Charnes, led to the introduction of the CCR Model (Charnes et al. 1978). The DEA technique was developed by Charnes, Banker and Cooper and the BCC Model was presented (Banker et al. 1984). The abovementioned models are reputedly known as the Classic DEA Models, with the objectives of computing efficiency, assuming systems as black boxes and neglecting the internal processes and interactions between them. In actual fact, the efficiency analysis in this method is carried out by the initial inputs and the final outputs. This aspect causes eradication of important information from the system, such as, its basis for inefficiency (Tone and Tsutsui 2009). This is in the condition that in the factual world, most of these systems have compound and complex structures, comprising of two or several stages and the performance of the components of these systems have an impact on the general efficiency (Cook et al. 2010; Akbari et al. 2020). For this purpose, Fare and Grosskopf (2000) introduced Network DEA Models (NDEA). These models defined the interactions and intermediate variables and similarly, by utilizing the series and parallel sub-divisions, dealt with evaluating the efficiency of complex systems (Chen and Yan 2011; An et al. 2019). Since the NDEA Models take into account the internal interactions of systems, hence, a more realistic performance of the systems can be demonstrated. In network models the performance of the entire system is calculated in relevance to the constraints or restrictions of the internal processes and the interactions between the general efficiency and that of the processes is established. Though, in the classic DEA Models, if the DMU has internal processes, the efficiency of these internal processes and the general process is computed independently and the correlations between the general efficiency and that of the processes is not conventional (Fukuyama and Weber 2010; Khalafi et al. 2021). Kao (2009) categorized the network models into three sets, namely, series, parallel and communicative. Kao stated that, when activities in a system are protracted in respect to each

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other, the system is of a series structure; and whenever activities are in a parallel form alongside each other, the system has a parallel structure. Similarly, when there is a hybrid condition between the series and parallel aspects, a communication mode is engaged. In order to calculate the efficiency of the entire network, both, in the series or parallel mode, usually, the efficiency coefficient attained in the stages relative to each other and the weighted average efficiency or the stages are normally and respectively utilized. In a series or parallel structure, a DMU is efficient when all its sub-processes are efficient (Shahriari 2013, Vaezi et al. 2020). Several studies have been carried out in relevance to NDEA and in respect to which, the task of Cook et al. (2010) can be indicted to in the year 2010. They developed a multi-stage model, in which each stage is able to consider the additional inputs and outputs. In fact, in this model, the outputs of each stage can be regarded as the final product and exit the system and or enter the next stage as an input. Thereby, each stage can take the additional inputs into consideration, as not being the outputs of the prior stage. In recent years, NDEA models have undergone development and models combining this science, with the game theory branch have been rendered. In another research, an et al. (2017) considered a network in two stages in an interactive mode and compared the efficiency of this network in a cooperative and non-cooperative (leader-follower mode). In yet another research by Zhou et al. (2018) who dealt with evaluating the efficiency performance of a multi-stage network in the black box and non-cooperative mode, comparing their results with each other. Other research in the grounds of network analysis that can be mentioned is the research relative to Du et al. (2015). They compared a parallel network in a cooperative and non-cooperative mode. On the basis of the abovementioned facts, the main difference between the black box and the network approach is summarized in the internal correlations of systems. Wang and Chin (2010) considered a two-stage model and denoted a relative weight for each stage. Next, they presented a model which converted the two stages to one stage and proved that, the overall efficiency of the two stages was equivalent to the efficiency of the single stage. This model is allegedly known as Wang's Model and its application in compound networks is to simplify the network. Actually, in Wang's Model, the approach is between the black box and the network; here, the internal interactions are somehow taken under consideration, but in order to simplify the performance, it is similar to the black box approach (Tavana et al. 2016; Vaezi 2021). In the recent years, special attention has been paid to undesirable factors in DEA Models. Such that, Liu et al. (2016) utilized the clustering methods and described this sphere as one of the four critical spheres or domains of DEA, from the researchers' viewpoint. Fare and Grosskopf (1989) for the initial time, mentioned the aspect of undesirable factors, in evaluating efficiency performance. Lu and Lo (2007) worked on undesirable outputs in the form of a sub-categorization and stated that, the initial method was to ignore the undesirable outputs. The second method was to restrict the extensiveness of the undesirable outputs or by taking the undesirable outputs into consideration as a nonlinear DEA model. The third method deliberates on the undesirable outputs, as being signified as inputs, or (by being marked) as negative outputs and or by imposing a single downward conversion. Over the past few years, Wang et al. (2013) and Wu et al. (2015) contemplated on the role of undesired factors in manufacturing processes and utilized the NDEA to measure efficiency. In recent years, the evolution of unfavorable features, has led to the use of undesirable factors for the generation of favorable aspects. For example, in a new approach, Wu et al. (2016) considered an interactive network consisting of two stages, where the first stage inserts the undesirable outputs to the second stage and ultimately, the second stage produces the desirable output and in actual fact, has utilized the undesirable outputs for production. The Data Envelopment Analysis with a double-frontier considers two efficiencies for each DMU. One is the optimistic view in which, each DMU, along with a set of efficient units, which form the efficient frontier, are compared. The other is the pessimistic view, where, each DMU, together with a set of inefficient units, forming the inefficient frontier are taken to comparison (Badiezadeh et al. 2018). The value of the optimistic view is less than or equal to (1); whereas, the efficiency of the pessimistic view is greater or equivalent to (1). The optimistic and pessimistic efficiency values are exactly equivalent to (1), if the DMU under evaluation, is placed respectively on the efficient or inefficient frontier (Azizi and Wang 2013; Vaezi 2022). In fact, the double-frontier views each DMU from two outlooks and any conclusion which implies to only one of these perspectives, shall be one-sided and inadequate. (Azizi 2012). Doyle et al. (1995) appraised and studied the performance from both, the optimistic and pessimistic viewpoint. In recent years, many researchers have utilized the double-frontier for evaluating efficiency and stated various approaches towards calculating an overall performance; in this concern tasks in relative to Wang et al. (2007) can be designated, who proposed a numerical measure for a general geometrical mean efficiency. Most of the research carried in the field of Data Envelopment Analysis is in static environments; and for the very first time, Sengupta (1995) performed efficiency evaluations in dynamic environments. In dynamic models, each time period is considered as a decision-making unit. Similarly, the correlation between the time periods in these models is contemplated by using additional inputs and outputs in between these periods (Jafarian and Ghoseiri 2011; Afzalinejad and Abbasi 2018). Since the era of Sengupta's tasks, till date, many articles have been published in the sphere of dynamic networks; the difference of which, lies in the case studies and the manner in which, the efficiency of the DMUs are computed. These include the Kawaguchi et al. (2014) and Wang et al. (2014) Models which can be indicated to in a dynamic mode respectively, for the evaluation of performance in hospital environments and banks. In accordance with the points mentioned, majority of the researches performed are network- concentrated and focused on two stages. Though, the current research takes a three-stage process under consideration, which, in addition to having intermediary variables, also has supplementary and undesirable inputs and outputs. We consider this network dynamic and deliberate on the optimistic and pessimistic approaches for its analysis. The purpose of this study is to calculate the efficiency of the entire network using the three perspectives which follow. The first outlook utilized for analysis in complex networks, due to its simplicity, is the black box. The second view is a standpoint which is in between the black box and the network analysis that also includes the intermediary variables, but has the black box approach. It is because of this, that we develop Wang's Method (2010) and utilize it for a three-stage process, taking it into consideration together with added and undesirable inputs and outputs. The third point of view concerns the cooperative mode. Due to the presence of additional inputs and outputs in the stages, a nonlinear model is utilized to solve this and we use a heuristic approach in this paper. In network analysis, internal activities of the system are focused on, but it is possible, that, due to the complexities of systems, modeling is not performed correctly. Similarly, majority of the NDEA Models are nonlinear and these two points could be the cause of bringing about a considerable amount of modification in efficiency results. Thereby, the objectives of this paper are to put together the results of the three abovementioned approaches, so as to achieve the final results. In continuation, the paper unfolds as follows: Section (2) describes the model and contemplates on the modeling of the three abovementioned approaches. In Section (3), the heuristic approach has been described, so as to resolve the cooperative view of the network analysis. Section (4) of the paper describes a factory, evaluating it dynamically and Section (5) concludes the paper.

2. Model description

We consider a set of n homogeneous decision making units (DMUs) that are denote by DMU_j (j=1,..., n), and each DMU_j (j=1,…,n) has three-stage, as shown in Fig. 1, where all the stages are connected together in series. We denote, the inputs of the first stage by $x^1_{i_1j}$ (i_1=1,…,I_1) and the undesirable outputs of the first stage by $y^1_{r_1j}$ (r_1=1,…,R_1). We denote, the intermediate measures between first stage and second stage by $z^1_{d_1j}$ (d_1=1,…,D_1) and between second stage and third stage by $z^2_{d_2j}$ (d_2=1,…,D_2). The additional inputs and outputs of the second stage are denoted by $x^2_{i_2j}$ (i_2=1,…,I_2) and $y^2_{r_2j}$ (r_2=1,…,R_2), respectively. Finally, we denote, the additional inputs of the third stage by $x^3_{i_3j}$ (i_3=1,…,I_3) and the outputs of the third stage by $y^3_{r_3j}$ (r_1=1,…,R_3). We adopt $v^1_{i_1}$, $v^2_{i_2}$ and $v^3_{i_3}$ as the weights of the inputs to the first, second and third stages, respectively. We adopt $w^1_{d_1}$ and $w^2_{d_2}$ as the weights of the intermediate measures between stage 1, stage 2 and stage 3, respectively. The weights of the outputs for the first, second and third stages second stage are denoted by $u^1_{r_1}$, $u^2_{r_2}$ and $u^3_{r_3}$, respectively.

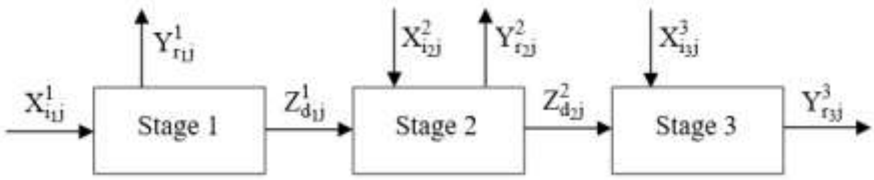

Figure1. Three-stage structure with additional inputs and outputs

To analyze the abovementioned network, we shall contemplate on three views, in relevance with the black box, a generalized model of Wang and a cooperative approach. In this section, we shall perform modeling for these three approaches respectively. Researchers in efficiency analysis are likely to use input-oriented models, due to three major reasons. Firstly, because demand is on the growth and estimating demand is an intricate matter. Secondly, managers have more control over inputs than outputs. Thirdly, this model reflects the primary goals of policymakers, based on being responsible in responding to the requirements of people and units must reduce costs, or else, limit the use of resources. Thereby, in this research we utilize the input-oriented model. In accordance with Korhonen and Luptacik (2004), we signify the undesirable outputs in the models with a negative mark.

2.1. Black box Approach

The black box approach is used to alleviate intricate networks and overlooks internal interactions and intermediary variables. The Figure 2, demonstrates a black box model for the network as shown in Figure 1, in which the inputs and outputs are three-stage inputs and outputs respectively.

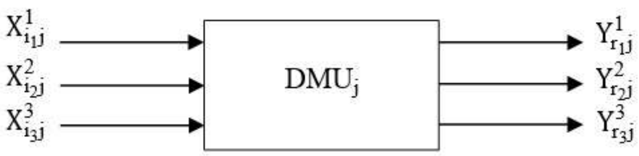

Figure 2. The black box models

We utilize an input-oriented CCR Model to evaluate the performance of the black box model. Hence, we define the maximal efficiency of the black box approach from the optimistic viewpoint, as follows:

$$\theta_o^{overall}=\max \sum_{r_2=1}^{R_2} u_{r_2}^2\, y_{r_2o}^2+\sum_{r_3=1}^{R_3} u_{r_3}^3\, y_{r_3o}^3-\sum_{r_1=1}^{R_1} u_{r_1}^1\, y_{r_1o}^1$$

$$\text{s.t. } \sum_{i_1=1}^{I_1} v_{i_1}^1\, x_{i_1o}^1+\sum_{i_2=1}^{I_2} v_{i_2}^2\, x_{i_2o}^2+\sum_{i_3=1}^{I_3} v_{i_3}^3\, x_{i_3o}^3=1 \qquad (1)$$

$$\sum_{R_2=1}^{R_2} u_{r_2}^2\, y_{r_2j}^2+\sum_{r_3=1}^{R_3} u_{r_3}^3\, y_{r_3j}^3-\sum_{r_1=1}^{R_1} u_{r_1}^1\, y_{r_1j}^1-\sum_{i_1=1}^{I_1} v_{i_1}^1\, x_{i_1j}^1-\sum_{i_2=1}^{I_2} v_{i_2}^2\, x_{i_2j}^2-$$

$$\sum_{i_3=1}^{I_3} v_{i_3}^3\, x_{i_3j}^3 \le 0$$

$$u_{r_1}^1,u_{r_2}^2,u_{r_3}^3 \ge \varepsilon;\ r_1=1,\dots,R_1;\ r_2=1,\dots,R_2;\ r_3=1,\dots,R_3;$$

$$v_{i_1}^1,v_{i_2}^2,v_{i_3}^3 \ge \varepsilon;\ i_1=1,\dots,I_1;\ \ i_2=1,\dots,I_2;\ \ i_3=1,\dots,I_3;\ j=1,\dots,n.$$

Jahed et al. (2015) evaluated the network performance by using an optimistic and pessimistic views. On the basis of the tasks of Jahed et al., we modified Model (1) and defined the efficiency of the black box approach from the pessimistic viewpoint as given below.

$$\varphi_o^{overall}=\min \sum_{r_2=1}^{R_2} u_{r_2}^2\, y_{r_2o}^2+\sum_{r_3=1}^{R_3} u_{r_3}^3\, y_{r_3o}^3-\sum_{r_1=1}^{R_1} u_{r_1}^1\, y_{r_1o}^1$$

$$\text{s.t. } \sum_{i_1=1}^{I_1} v_{i_1}^1\, x_{i_1o}^1+\sum_{i_2=1}^{I_2} v_{i_2}^2\, x_{i_2o}^2+\sum_{i_3=1}^{I_3} v_{i_3}^3\, x_{i_3o}^3=1 \qquad (2)$$

$$\sum_{R_2=1}^{R_2} u_{r_2}^2\, y_{r_2j}^2+\sum_{r_3=1}^{R_3} u_{r_3}^3\, y_{r_3j}^3-\sum_{r_1=1}^{R_1} u_{r_1}^1\, y_{r_1j}^1-\sum_{i_1=1}^{I_1} v_{i_1}^1\, x_{i_1j}^1-\sum_{i_2=1}^{I_2} v_{i_2}^2\, x_{i_2j}^2-$$

$$\sum_{i_3=1}^{I_3} v_{i_3}^3\, x_{i_3j}^3 \ge 0$$

$$u_{r_1}^1,u_{r_2}^2,u_{r_3}^3 \ge \varepsilon;\ r_1=1,\dots,R_1;\ r_2=1,\dots,R_2;\ r_3=1,\dots,R_3;$$

$v_{i_1}^1,v_{i_2}^2,v_{i_3}^3 \ge \varepsilon;\ i_1=1,\dots,I_1;\ \ i_2=1,\dots,I_2;\ \ i_3=1,\dots,I_3;\ j=1,\dots,n.$ Wang and Chin (2009) utilized the double-frontier for efficiency appraisal and they proposed a geometric mean to compute a general performance. Thereby, we define the efficiency of the black box approach, by taking the double-frontier into consideration, on the results of Models (1) and (2) according to the following:

$$\emptyset_o=\sqrt{\theta_o^{overall}.\varphi_o^{overall}} \qquad (3)$$

2.2. Wang's Approach (2010)

One of the models of the network, which considers the intermediary products of the classic models and to a certain extent, has covered the void in these models (Wang Model 2010). Wang took a two-stage structure, without additional inputs and outputs into consideration and for each stage assumed λ_1 and λ_2 as relative weights. Wang defined the total efficiency as $\theta_o = \lambda_1\theta_o^1+\lambda_2\theta_o^2$ and the values of λ_i was in accordance with the views of managers in relative to$\lambda_1 + \lambda_2 = 1$. In Wang's model, we extend this model into three levels to evaluate efficiency Fig. 1, by contemplating the additional inputs and outputs. In order to utilize Wang's model, we initially focus on the chain which comprises of the first and second stages and by taking advantage of Wang's model (according to Figure 3) we convert it into a single stage.

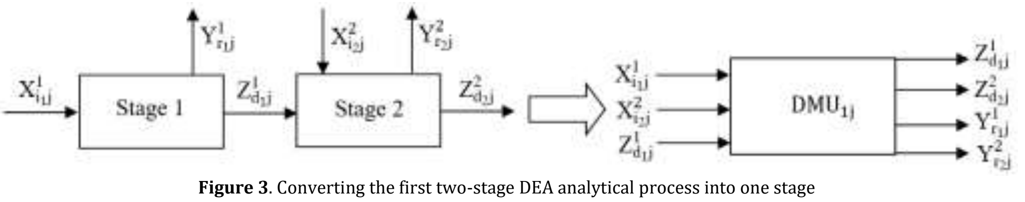

Figure 3. Converting the first two-stage DEA analytical process into one stage

As formerly discussed, we take into consideration the second part of the chain (Figure 1) and by utilizing the Wang Model; we convert it into a stage according to Figure 4.

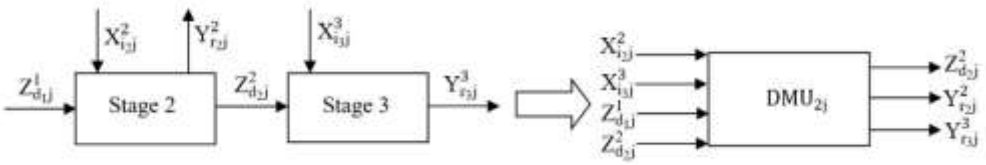

Figure 4. Converting the second two-stage DEA analytical process into one stage

In considering figures (3) and (4), the three-stage DEA process, in accordance with Figure. 5, has been reduced to a two-stage process by the Wang Model.

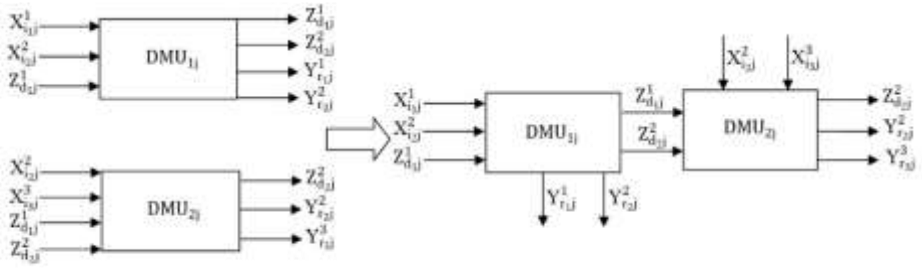

Figure 5. Converting the three-stage DEA analytical process to a two-stage process

Finally, by performing another Wang conversion, Figure. 5, according to the following figure, it is converted into a single stage. The Fig. 6 demonstrates the conversion of the three-stage process to a single stage by utilizing the Wang Model.

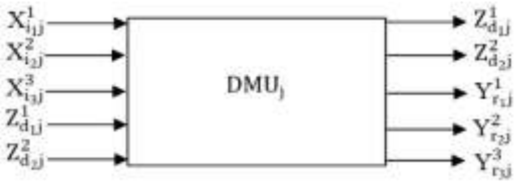

Figure 6. Converting the three-stage DEA analytical process to a single stage

For every stage Fig. 5, we considered λ_1 and λ_2 as relative weights. Hence, the maximal efficiency demonstrated by the network in (Fig. 1) and by using a generalized approach of Wang, we define the optimistic view as hereunder:

$$
\begin{aligned}
\theta_o^{overall} = \max\ & \lambda_1\Big(\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2o}^2 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1o}^1\Big) + \\
& \lambda_2\Big(\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2o}^2 + \sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3o}^3\Big) \qquad (4) \\
\text{s.t.}\ & \lambda_1\Big(\sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1o}^1 + \sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2o}^2 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1\Big) + \\
& \lambda_2\Big(\sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2o}^2 + \sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3o}^3 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2\Big) = 1 \\
& \sum_{D_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2j}^2 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1j}^1 - \sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1j}^1 - \\
& \quad \sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2j}^2 - \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 \leq 0, \quad j=1,\dots,n \\
& \sum_{D_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2j}^2 + \sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3j}^3 - \sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2j}^2 - \sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3j}^3 - \\
& \quad \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 - \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 \leq 0, \quad j=1,\dots,n \\
& u_{r_1}^1, u_{r_2}^2, u_{r_3}^3 \geq \varepsilon;\ r_1=1,\dots,R_1;\ r_2=1,\dots,R_2;\ r_3=1,\dots,R_3; \\
& v_{i_1}^1, v_{i_2}^2, v_{i_3}^3 \geq \varepsilon;\ i_1=1,\dots,I_1;\ \ i_2=1,\dots,I_2;\ \ i_3=1,\dots,I_3; \\
& \quad w_{d_1}^1, w_{d_2}^2 \geq \varepsilon;\ d_1=1,\dots,D_1;\ \ d_2=1,\dots,D_2.
\end{aligned}
$$

In accordance with the tasks of Wang and Chin, (2009), we have modified Model (4) and the minimal pessimistic efficiency Figure. 1 has been defined from the Wang's generalized approach, as given below:

$$\begin{aligned}
\varphi_o^{overall} = \min\ & \lambda_1\Big(\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2o}^2 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1o}^1\Big) + \\
& \lambda_2\Big(\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2o}^2 + \sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3o}^3\Big) \qquad (5)\\
\text{s.t.}\ & \lambda_1\Big(\sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1o}^1 + \sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2o}^2 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1\Big) + \\
& \lambda_2\Big(\sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2o}^2 + \sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3o}^3 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2\Big) = 1\\
& \sum_{D_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2j}^2 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1j}^1 - \sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1j}^1 - \\
& \quad \sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2j}^2 - \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 \geq 0, \quad j=1,\ldots,n\\
& \sum_{D_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2j}^2 + \sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3j}^3 - \sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2j}^2 - \sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3j}^3 - \\
& \quad \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 - \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 \geq 0, \quad j=1,\ldots,n\\
& u_{r_1}^1, u_{r_2}^2, u_{r_3}^3 \geq \varepsilon;\ r_1=1,\ldots,R_1;\ r_2=1,\ldots,R_2;\ r_3=1,\ldots,R_3;\\
& v_{i_1}^1, v_{i_2}^2, v_{i_3}^3 \geq \varepsilon;\ i_1=1,\ldots,I_1;\ \ i_2=1,\ldots,I_2;\ \ i_3=1,\ldots,I_3;\\
& w_{d_1}^1, w_{d_2}^2 \geq \varepsilon;\ d_1=1,\ldots,D_1;\ \ d_2=1,\ldots,D_2.
\end{aligned}$$

Models (4) and (5) are linear models and the values of λ_1 and λ_2 , by taking into account the correlation that $\lambda_1 + \lambda_2 = 1$ which is determined by the opinions of experts. After solving models (4) and (5), we utilize formula (3) to calculate the generalized efficiency of Wang's approach, which is computed by contemplating on the double-frontier.

2.3 The Cooperative Approach

One of the methods of the network analysis is the cooperative approach, where all the sub-units make attempts to elevate the efficiency of the network to the maximal. Due to this, the network illustrated in Fig. 1 is comprised of three sub-units and the efficiency of these sub-units is calculated as given hereunder.

$$\begin{aligned}
\theta_o^1 = \max\ & \frac{\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1o}^1}{\sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1o}^1}\\
\text{s.t.}\ & \frac{\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1j}^1}{\sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1j}^1} \leq 1 \quad , \quad j=1,\ldots,n \qquad (6)\\
& u_{r_1}^1, v_{i_1}^1, w_{d_1}^1 \geq \varepsilon;\ r_1=1,\ldots,R_1;\ i_1=1,\ldots,I_1;\ d_1=1,\ldots,D_1.
\end{aligned}$$

$$\begin{aligned}
\theta_o^2 = \max\ & \frac{\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2o}^2}{\sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2o}^2 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1}\\
\text{s.t.}\ & \frac{\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2j}^2}{\sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2j}^2 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1} \leq 1 \ , \ j=1,\ldots,n \qquad (7)\\
& u_{r_2}^2, v_{i_2}^2, w_{d_1}^1, w_{d_2}^2 \geq \varepsilon;\ r_2=1,\ldots, R_2;\ \ i_2=1,\ldots,I_2;\ d_1=1,\ldots,D_1;\ \ d_2=1,\ldots,D_2.
\end{aligned}$$

$$\theta_o^3 = \max\ \frac{\sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3o}^3}{\sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3o}^3 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2}$$

$$\text{s.t.}\quad \frac{\sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3j}^3}{\sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3j}^3 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2} \leq 1 \quad , \quad j=1,\dots,n \tag{8}$$

$$u_{r_3}^3, v_{i_3}^3, w_{d_2}^2 \geq \varepsilon;\ r_3=1,\dots,R_3;\ i_3=1,\dots,I_3;\ d_2=1,\dots,D_2.$$

Kao and Hwang (2008) used uniform weights for the intermediate variables to evaluate the efficiency of a network. Therefore, we utilized similar weights for the intermediate variables in models (6), (7) and (8). For the network structure as shown in Fig.1, the first, second and the third stages are linked in series. Kao and Hwang (2008) used the multiplicative approach to measure the overall efficiency of a series structure. We then define the overall efficiency of an integrated system shown in Fig.1 as $\theta_o^{overall}=\max \theta_o^1 . \theta_o^2 . \theta_o^3$ Thus:

$$\theta_o^{overall}=\max \frac{\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1o}^1}{\sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1o}^1} . \frac{\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2o}^2}{\sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2o}^2 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1} . \frac{\sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3o}^3}{\sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3o}^3 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2}$$

$$\text{s.t.}\quad \frac{\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1j}^1}{\sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1j}^1} \leq 1, \quad j=1,\dots,n$$

$$\frac{\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2j}^2}{\sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2j}^2 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1} \leq 1, \quad j=1,\dots,n \tag{9}$$

$$\frac{\sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3j}^3}{\sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3j}^3 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2} \leq 1, \quad j=1,\dots,n$$

$$u_{r_1}^1, u_{r_2}^2, u_{r_3}^3 \geq \varepsilon;\ r_1=1,\dots,R_1;\ r_2=1,\dots,R_2;\ r_3=1,\dots,R_3;$$

$$v_{i_1}^1, v_{i_2}^2, v_{i_3}^3 \geq \varepsilon;\ i_1=1,\dots,I_1;\quad i_2=1,\dots,I_2;\quad i_3=1,\dots,I_3;$$

$$w_{d_1}^1, w_{d_2}^2 \geq \varepsilon;\ d_1=1,\dots,D_1;\quad d_2=1,\dots,D_2.$$

Model (9) demonstrates the maximal overall efficiency of the network in Fig. 1; and measures it from the optimistic view, on condition that the efficiency of all the stages is less than (1). We, based on the tasks of Wang and Chin, (2009), have modified Model (9) and have defined the minimal overall pessimistic efficiency (Fig. 1) from the co-operative approach according to the following:

$$\varphi_o^{overall}=\min \frac{\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1o}^1}{\sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1o}^1} . \frac{\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2o}^2}{\sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2o}^2 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1} . \frac{\sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3o}^3}{\sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3o}^3 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2}$$

$$\text{s.t.}\quad \frac{\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1j}^1}{\sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1j}^1} \geq 1, \quad j=1,\dots,n$$

$$\frac{\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2j}^2}{\sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2j}^2 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1} \geq 1, \quad j=1,\dots,n \tag{10}$$

$$\frac{\sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3j}^3}{\sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3j}^3 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2} \geq 1, \quad j=1,\dots,n$$

$$u_{r_1}^1, u_{r_2}^2, u_{r_3}^3 \geq \varepsilon;\ r_1=1,\dots,R_1;\ r_2=1,\dots,R_2;\ r_3=1,\dots,R_3;$$

$$v_{i_1}^1, v_{i_2}^2, v_{i_3}^3 \geq \varepsilon;\ i_1=1,\dots,I_1;\quad i_2=1,\dots,I_2;\quad i_3=1,\dots,I_3;$$

$$w_{d_1}^1, w_{d_2}^2 \geq \varepsilon;\ d_1=1,\dots,D_1;\quad d_2=1,\dots,D_2.$$

Models (9) and (10) are nonlinear models and in the third section of this paper an innovative approach is used to solve them. In assuming that models (9 and 10) are resolved, we utilize formula (3) to compute the cooperative efficiency of the network by taking the double-frontier into consideration. So, we performed the modeling of the network shown in Fig 1, from three approaches: Black box, Wang and Cooperative Approaches. In the real world, decision-makers do not confine themselves to one method for decision-making purposes; and it is possible that, by utilizing various other methods, achieve several results. In such conditions, skills to put together results of techniques have been proposed. One of these is the Borda Method. When using different methods for ranking, varied results are obtained and the Borda Method is taken advantage of, to attain a single ranking. This method is founded on the unanimous basis and the options are ranked in a paired comparison manner. In order to rank units by utilizing the Borda Method, if the row is superior to the column it is signified by “M” and if the column is greater than the row it is denoted as “X”. The “O” symbol also means the equality of the row and column. Ultimately, the total of each Borda row is inserted into the column and the row with the highest win (highest numerical for M) secures the utmost ranking. In this paper, we use the Borda technique to integrate the results of the described approaches and these results are shown in the case study section.

3. Heuristic approach to solve the cooperative view

Due to the presence of additional inputs and outputs in the stages, models (9 and 10) are nonlinear. In order to solve this model, we use a heuristic approach as given hereunder:

3.1. Heuristic approach to solve the optimistic view

We are aware that the objective function of model (9) is the product of the multiplicative efficiency of the three-stage process i.e. $\theta_o^{overall} = \max \theta_o^1 . \theta_o^2 . \theta_o^3$. We consider θ_o^1 and θ_o^2 as variables in the objective function, which are between the $[0, \theta_o^{1\text{-max}}]$ and $[0, \theta_o^{2\text{- max}}]$ intervals and change respectively. We describe θ_o^1 and θ_o^2 in the figure below, so that we can move them within the intervals.

$$\theta_o^1 = \theta_o^{1\text{-max}} - k_1\Delta\varepsilon, \quad k_1=0,1,\ldots,\left[\frac{\theta_o^{1\text{- max}}}{\Delta\varepsilon}\right]+1 \tag{11}$$

$$\theta_o^2 = \theta_o^{2\text{-max}} - k_2\Delta\varepsilon, \quad k_2=0,1,\ldots,\left[\frac{\theta_o^{2\text{- max}}}{\Delta\varepsilon}\right]+1$$

In the formula (11), we consider $\Delta\varepsilon$ as a step size and of a very small value; and define $\theta_o^{1\text{-max}}$ and $\theta_o^{2\text{-max}}$ respectively, as the maximum optimistic efficiency of stages (1 and 2) in Fig. 1. From the models rendered hereunder, they are capable of being calculated.

$$\theta_o^{1\text{-max}}=\max\left\{\theta_o^1 \middle| \theta_j^1 \leq 1, \theta_j^2 \leq 1, \theta_j^3 \leq 1, \ j=1,\ldots,n\right\} \tag{12}$$

$$\theta_o^{2\text{-max}}=\max\left\{\theta_o^2 \middle| \theta_j^1 \leq 1, \theta_j^2 \leq 1, \theta_j^3 \leq 1, \ j=1,\ldots,n\right\}$$

All the variables are non-negative in model (12). The said models have attained a maximum efficiency in the first and second stages, on condition that, the efficiency of the stages is less than (1). These models are fractions and by utilizing the Charnes-Cooper conversion (1962), such as, given below, they are modified into linear models.

$$\begin{aligned}
\theta_o^{1\text{-max}} = \max\ & \textstyle\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1o}^1 \\
\text{s.t.}\ & \textstyle\sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1o}^1 = 1 \\
& \textstyle\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1j}^1 - \sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1j}^1 \leq 0 \\
& \textstyle\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2j}^2 - \sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2j}^2 - \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 \leq 0 \\
& \textstyle\sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3j}^3 - \sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3j}^3 - \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 \leq 0 \\
& u_{r_1}^1, u_{r_2}^2, u_{r_3}^3 \geq \varepsilon;\ r_1=1,\ldots,R_1;\ r_2=1,\ldots,R_2;\ r_3=1,\ldots,R_3; \\
& v_{i_1}^1, v_{i_2}^2, v_{i_3}^3 \geq \varepsilon;\ i_1=1,\ldots,I_1;\ \ i_2=1,\ldots,I_2;\ \ i_3=1,\ldots,I_3; \\
& w_{d_1}^1, w_{d_2}^2 \geq \varepsilon;\ d_1=1,\ldots,D_1;\ \ d_2=1,\ldots,D_2.
\end{aligned} \tag{13}$$

$$\begin{aligned}
\theta_o^{2\text{-max}} = \max\ & \textstyle\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2o}^2 \\
\text{s.t.}\ & \textstyle\sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2o}^2 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 = 1
\end{aligned}$$

$$\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1j}^1 - \sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1j}^1 \le 0 \tag{14}$$

$$\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2j}^2 - \sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2j}^2 - \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 \le 0$$

$$\sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3j}^3 - \sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3j}^3 - \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 \le 0$$

$$u_{r_1}^1, u_{r_2}^2, u_{r_3}^3 \ge \varepsilon;\ r_1=1,\ldots,R_1;\ r_2=1,\ldots,R_2;\ r_3=1,\ldots,R_3;$$

$$v_{i_1}^1, v_{i_2}^2, v_{i_3}^3 \ge \varepsilon;\ i_1=1,\ldots,I_1;\ \ i_2=1,\ldots,I_2;\ \ i_3=1,\ldots,I_3;$$

$$w_{d_1}^1, w_{d_2}^2 \ge \varepsilon;\ d_1=1,\ldots,D_1;\ \ d_2=1,\ldots,D_2.$$

In determining the value of $\theta_o^{1\text{-max}}$ and $\theta_o^{2\text{-max}}$ with the assistance of models (13) and (14), we alter model (9) and convert it to the following model.

$$\theta_o^{\text{overall}} = \max \left\{ \theta_o^1 . \theta_o^2 . \theta_o^3 \;\middle|\; \begin{array}{l} \theta_j^1 \le 1,\ \theta_j^2 \le 1,\ \theta_j^3 \le 1, \theta_o^1 = \frac{O_o^1}{I_o^1},\ \theta_o^2 = \frac{O_o^2}{I_o^2}, \\ \theta_o^1 \in [0,\ \theta_o^{1\text{-max}}],\ \theta_o^2 \in [0,\ \theta_o^{2\text{-max}}],\ j=1,\ldots,n \end{array} \right\} \tag{15}$$

It should be observed that in the model (15), we consider θ_o^1 and θ_o^2 in the objective function as two variables and two constraints which specify these two variables and together with its interval modifications, it was supplemented to the model. In models (6) and (7), we have described the efficiencies of stages (1) and (2) and in the model (15) have briefly illustrated it in the form of outputs and inputs of each stage or $\theta_o^1 = \frac{O_o^1}{I_o^1}$ and $\theta_o^2 = \frac{O_o^2}{I_o^2}$. The model (15) is a fractional model and by utilizing the Charnes-Cooper conversion (1962), such as, given below, is modified into linear models.

$$\theta_o^{\text{overall}} = \max\ \theta_o^1 . \theta_o^2 . \sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3o}^3$$

$$\text{s.t.}\ \ \sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3o}^3 + \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2 = 1$$

$$\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1j}^1 - \sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1j}^1 \le 0 \tag{16}$$

$$\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2j}^2 - \sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2j}^2 - \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 \le 0$$

$$\sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3j}^3 - \sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3j}^3 - \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 \le 0$$

$$\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1o}^1 - \theta_o^1 \sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1o}^1 = 0$$

$$\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2o}^2 - \theta_o^2 \left(\sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2o}^2 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 \right) = 0$$

$$\theta_o^1 \in [0,\ \theta_o^{1\text{-max}}]$$

$$\theta_o^2 \in [0,\ \theta_o^{2\text{-max}}]$$

$$u_{r_1}^1, u_{r_2}^2, u_{r_3}^3 \ge \varepsilon;\ r_1=1,\ldots,R_1;\ r_2=1,\ldots,R_2;\ r_3=1,\ldots,R_3;$$

$$v_{i_1}^1, v_{i_2}^2, v_{i_3}^3 \ge \varepsilon;\ i_1=1,\ldots,I_1;\ \ i_2=1,\ldots,I_2;\ \ i_3=1,\ldots,I_3;$$

$$w_{d_1}^1, w_{d_2}^2 \ge \varepsilon;\ d_1=1,\ldots,D_1;\ \ d_2=1,\ldots,D_2.$$

In model (16), by utilizing formula (11), we increase the values of k1 and k2 independently, from (0) to a high level for each one, so that each time the model can be solved with the new θ_o^1 and θ_o^2. We resolve all the returns of the conditions of the k1 and k2 models and illustrate the responses with $\theta_o^{\text{overall}}(k_1, k_2)$. By comparing the overall values of $\theta_o^{\text{overall}}(k_1, k_2)$, we describe the maximal efficiency of $\theta_o^{\text{overall}}(k_1, k_2)$ in Fig. 1 from the optimistic view. It should be noted that, we have tested our proposed approach in three modes and each time have taken two stages into consideration as variables. With due attention to this point that, the efficiency of Fig. 1 is unique, hence, the results of these three methods are remarkably in approximation to each other and in order to explain we have raised one of these three conditions to describe our above approach.

3.2. Heuristic approach to solve the pessimistic view

Our analogous optimistic approach φ_o^1 and φ_o^2 are considered as two variables in the objective function model (10), which changes respectively, between intervals $[\varphi_o^{1\text{-min}}, M]$ and $[\varphi_o^{2\text{-min}}, M]$. We describe φ_o^1 and φ_o^2 in the following manner, so that we can move them within the intervals.

$$\varphi_o^1 = \varphi_o^{1\text{-min}} + k_1\Delta\varepsilon, \quad k_1=0,1,\ldots,\left[\frac{M-\varphi_o^{1\text{-min}}}{\Delta\varepsilon}\right]+1 \tag{17}$$

$$\varphi_o^2 = \varphi_o^{2\text{-min}} + k_2\Delta\varepsilon, \quad k_2=0,1,\ldots,\left[\frac{M-\varphi_o^{2\text{-min}}}{\Delta\varepsilon}\right]+1$$

We take "M" as a larger value and $\Delta\varepsilon$ as a similar optimistic approach, as a step size and of very small value. Moreover, $\varphi_o^{1\text{-min}}$ and $\varphi_o^{2\text{-min}}$ are respectively, the minimal optimistic efficiency of the first and second stages, which has been described in Fig. 1 and from the following models their values are capable of being computed.

$$\varphi_o^{1\text{-min}}=\min\left\{\varphi_o^1 \middle| \ \varphi_j^1 \geq 1,\ \varphi_j^2 \geq 1,\ \varphi_j^3 \geq 1,\ \ j=1,\ldots,n\right\} \tag{18}$$

$$\varphi_o^{2\text{-min}}=\min\left\{\varphi_o^2 \middle| \ \varphi_j^1 \geq 1,\ \varphi_j^2 \geq 1,\ \varphi_j^3 \geq 1,\ \ j=1,\ldots,n\right\}$$

The entire variables are non-negative in the model (18). The said models have attained a minimum efficiency in the first and second stages, on condition that, the efficiency of the stages is more than (1). These models are fractions and by utilizing the Charnes-Cooper conversion (1962), such as, given below, they are modified into linear models.

$$\begin{aligned}
\varphi_o^{1\text{-min}} = \min\ & \textstyle\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1o}^1 \\
\text{s.t.}\ \ & \textstyle\sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1o}^1 = 1 \\
& \textstyle\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1j}^1 - \sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1j}^1 \geq 0 \\
& \textstyle\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2j}^2 - \sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2j}^2 - \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 \geq 0 \\
& \textstyle\sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3j}^3 - \sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3j}^3 - \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 \geq 0 \\
& u_{r_1}^1, u_{r_2}^2, u_{r_3}^3 \geq \varepsilon;\ r_1=1,\ldots,R_1;\ r_2=1,\ldots,R_2;\ r_3=1,\ldots,R_3; \\
& v_{i_1}^1, v_{i_2}^2, v_{i_3}^3 \geq \varepsilon;\ i_1=1,\ldots,I_1;\ \ i_2=1,\ldots,I_2;\ \ i_3=1,\ldots,I_3; \\
& w_{d_1}^1, w_{d_2}^2 \geq \varepsilon;\ d_1=1,\ldots,D_1;\ \ d_2=1,\ldots,D_2.
\end{aligned} \tag{19}$$

$$\begin{aligned}
\varphi_o^{2\text{-min}} = \min\ & \textstyle\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2o}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2o}^2 \\
\text{s.t.}\ \ & \textstyle\sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2o}^2 + \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1o}^1 = 1 \\
& \textstyle\sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1 y_{r_1j}^1 - \sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1j}^1 \geq 0 \\
& \textstyle\sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2 y_{r_2j}^2 - \sum_{i_2=1}^{I_2} v_{i_2}^2 x_{i_2j}^2 - \sum_{d_1=1}^{D_1} w_{d_1}^1 z_{d_1j}^1 \geq 0 \\
& \textstyle\sum_{r_3=1}^{R_3} u_{r_3}^3 y_{r_3j}^3 - \sum_{i_3=1}^{I_3} v_{i_3}^3 x_{i_3j}^3 - \sum_{d_2=1}^{D_2} w_{d_2}^2 z_{d_2j}^2 \geq 0 \\
& u_{r_1}^1, u_{r_2}^2, u_{r_3}^3 \geq \varepsilon;\ r_1=1,\ldots,R_1;\ r_2=1,\ldots,R_2;\ r_3=1,\ldots,R_3; \\
& v_{i_1}^1, v_{i_2}^2, v_{i_3}^3 \geq \varepsilon;\ i_1=1,\ldots,I_1;\ \ i_2=1,\ldots,I_2;\ \ i_3=1,\ldots,I_3; \\
& w_{d_1}^1, w_{d_2}^2 \geq \varepsilon;\ d_1=1,\ldots,D_1;\ \ d_2=1,\ldots,D_2.
\end{aligned} \tag{20}$$

By determining the values of $\varphi_o^{1\text{-min}}$ and $\varphi_o^{2\text{-min}}$ with the help of models (19) and (20), we convert model (10) to model (21) as follows:

$$\varphi_o^{overall}=\min\left\{\varphi_o^1 .\varphi_o^2 . \varphi_o^3 \;\middle|\; \begin{array}{l}\varphi_j^1 \geq 1, \varphi_j^2 \geq 1, \varphi_j^3 \geq 1, \varphi_o^1 = \frac{O_o^1}{I_o^1}, \varphi_o^2 = \frac{O_o^2}{I_o^2}, \\ \varphi_o^1 \in [\varphi_o^{1\text{-min}}, M], \varphi_o^2 \in [\varphi_o^{2\text{-min}}, M], j=1,\ldots,n\end{array}\right\} \tag{21}$$

It should be noted that similar to the optimistic approach in the model (21), we consider φ_o^1 and φ_o^2 in the objective function as two variables and two constraints which specify these two variables and together with its interval modifications, it was supplemented to the model. The model (21) is a fractional one and by utilizing the Charnes-Cooper conversion (1962), such as, given below, they are modified into a linear model.

$$\begin{aligned}
\varphi_o^{overall} = \min\ & \varphi_o^1 .\varphi_o^2 . \textstyle\sum_{r_3=1}^{R_3} u_{r_3}^3\, y_{r_3o}^3 \\
\text{s.t.}\ & \textstyle\sum_{i_3=1}^{I_3} v_{i_3}^3\, x_{i_3o}^3 + \sum_{d_2=1}^{D_2} w_{d_2}^2\, z_{d_2o}^2 = 1 \\
& \textstyle\sum_{d_1=1}^{D_1} w_{d_1}^1\, z_{d_1j}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1\, y_{r_1j}^1 - \sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1j}^1 \geq 0 \\
& \textstyle\sum_{d_2=1}^{D_2} w_{d_2}^2\, z_{d_2j}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2\, y_{r_2j}^2 - \sum_{i_2=1}^{I_2} v_{i_2}^2\, x_{i_2j}^2 - \sum_{d_1=1}^{D_1} w_{d_1}^1\, z_{d_1j}^1 \geq 0 \\
& \textstyle\sum_{r_3=1}^{R_3} u_{r_3}^3\, y_{r_3j}^3 - \sum_{i_3=1}^{I_3} v_{i_3}^3\, x_{i_3j}^3 - \sum_{d_2=1}^{D_2} w_{d_2}^2\, z_{d_2j}^2 \geq 0 \\
& \textstyle\sum_{d_1=1}^{D_1} w_{d_1}^1\, z_{d_1o}^1 - \sum_{r_1=1}^{R_1} u_{r_1}^1\, y_{r_1o}^1 - \varphi_o^1 \sum_{i_1=1}^{I_1} v_{i_1}^1 x_{i_1o}^1 = 0 \\
& \textstyle\sum_{d_2=1}^{D_2} w_{d_2}^2\, z_{d_2o}^2 + \sum_{r_2=1}^{R_2} u_{r_2}^2\, y_{r_2o}^2 - \varphi_o^2 \left(\sum_{i_2=1}^{I_2} v_{i_2}^2\, x_{i_2o}^2 + \sum_{d_1=1}^{D_1} w_{d_1}^1\, z_{d_1o}^1\right) = 0 \\
& \varphi_o^1 \in [\varphi_o^{1\text{-min}}, M] \\
& \varphi_o^2 \in [\varphi_o^{2\text{-min}}, M] \\
& u_{r_1}^1, u_{r_2}^2, u_{r_3}^3 \geq \varepsilon;\ r_1=1,\ldots,R_1;\ r_2=1,\ldots,R_2;\ r_3=1,\ldots,R_3; \\
& v_{i_1}^1, v_{i_2}^2, v_{i_3}^3 \geq \varepsilon;\ i_1=1,\ldots,I_1;\ \ i_2=1,\ldots,I_2;\ \ i_3=1,\ldots,I_3; \\
& w_{d_1}^1, w_{d_2}^2 \geq \varepsilon;\ d_1=1,\ldots,D_1;\ \ d_2=1,\ldots,D_2.
\end{aligned} \tag{22}$$

In model (22) by utilizing formula (17), the values of k1 and k2 are increased independently from (0) to a high level for each one, so that, we can resolve the model with the new φ_o^1 and φ_o^2 . We solve all the returns of the conditions of the k1 and k2 models and show the responses with $\varphi_o^{overall}(k_1, k_2)$. By comparing the overall values of $\varphi_o^{overall}(k_1, k_2)$, we describe the minimal efficiency of $\varphi_o^{overall}(k_1, k_2)$ in Fig. 1 from the pessimistic view. It should be observed that, we have tested our proposed approach in three modes and each time have taken two stages into consideration as variables. Given that at this point, the efficiency of Fig. 1 is unique, hence, the results of these three methods are outstandingly in approximation to each other and in order to explain, we have broached one of these three conditions to describe our above approach.

4. Case study description

The Data Envelopment Analysis (DEA) is a theoretical framework which discusses the analyzing of efficiency and its application in the arena of production planning and inventory control is observed to a very slight extent. In this paper, an example in the field of production planning and inventory control has been described, from the real world, as follows. Let us consider a dairy factory which produces three products. This factory has a production area, warehouse premises and a delivery point. We consider each of these as a stage and take this factory into consideration as a dynamic network for duration of 24 time periods. In this network, a number of outputs in time period t in the second stage are modified into a number of inputs to the second stage during the time period t+1. We assume each time period as a DMU. Hence, the inputs and outputs of each DMU are according to the following explanations. We consider the production costs of three products as an input to the first stage and signify it as (x_1^1, x_2^1, x_3^1). We describe the transport cost of produce from the first to the second stage as an undesirable output of stage (1), and illustrate it as (y_1^1). The intermediary produce between the first and the second stage is the quantity of production for each commodity and this shown by (z_1^1, z_2^1, z_3^1). The additional inputs to the second stage are respectively, the cost of reserving storage location (x_1^2), cost of holding goods (x_2^2) and the goods remaining from the previous period in the warehouse, which is illustrated as (x_3^2, x_4^2, x_5^2). We shall describe the output of the second stage as the quantity of goods remaining in the warehouse for the next period of time and this is demonstrated as (y_1^2, y_2^2, y_3^2). The intermediary products between the second and the third stages is the quantity of delivery of each commodity, is shown as (z_1^2, z_2^2, z_3^2). The additional inputs of the third stage are the transport of goods to this stage, which is demonstrated as (x_1^3) and finally, (y_1^3) denotes the outputs of the third stage which are the profits gained from the sale of goods. In continuation, we have demonstrated the number of inputs for a time period of 24 intervals in Table (1) and the intermediary amounts and outputs in Table (2).

Table 1. The inputs of the factory for 24 period in 2016

DMU	Production cost			Cost of reserving storage location	Cost of holding goods	Goods remaining from the previous period			Cost of Transport goods to delivery points
	x_1^1	x_2^1	x_3^1	x_1^2	x_2^2	x_3^2	x_4^2	x_5^2	x_1^3
1	29120000	36160000	51520000	1700000	1430000	0	0	0	3680000
2	50960000	63280000	77280000	1700000	1430000	0	0	0	6235000
3	80080000	99440000	128800000	1700000	1430000	0	0	0	9915000
4	101920000	126560000	180320000	1700000	1430000	0	0	0	12880000
5	43680000	54240000	77280000	1700000	1430000	0	0	0	5520000
6	50960000	63280000	103040000	1700000	1430000	0	0	0	6645000
7	94640000	126560000	154560000	1700000	1670000	0	0	0	11755000
8	145600000	180800000	257600000	1700000	3620000	0	2	0	15435000
9	145600000	180800000	257600000	1700000	3170000	6	8	4	19115000
10	145600000	180800000	257600000	1700000	1730000	4	6	4	20555000
11	145600000	180800000	257600000	1700000	1430000	0	0	2	19220000
12	145600000	180800000	257600000	1700000	1430000	0	0	0	16815000
13	87360000	99440000	128800000	1700000	1430000	0	0	0	10290000
14	50960000	63280000	77280000	1700000	1430000	0	0	0	6235000
15	50960000	63280000	103040000	1700000	1430000	0	0	0	6645000
16	43680000	54240000	77280000	1700000	1430000	0	0	0	5520000
17	80080000	99440000	128800000	1700000	1430000	0	0	0	9915000
18	94640000	117520000	154560000	1700000	1430000	0	0	0	11755000
19	72800000	90400000	128800000	1700000	1430000	0	0	0	9200000
20	87360000	108480000	154560000	1700000	1430000	0	0	0	11040000
21	87360000	108480000	128800000	1700000	1430000	0	0	0	10630000
22	109200000	135600000	180320000	1700000	3830000	0	0	0	9915000
23	145600000	180800000	257600000	1700000	1430000	8	8	4	22080000
24	145600000	180800000	257600000	1700000	1430000	0	0	0	18400000

In the Table 1, the (0) values for each time period indicates that the goods have not remained in the warehouse from the prior period (columns 7 to 9). The Table below (Table 2), also expresses the fact that the (0) values signify that no goods have remained in the warehouse for the subsequent period (columns 9 to 11).

Table 2. The outputs and the intermediate measures of the factory for 24 period in 2016

DMU	Quantity of each goods produced			Quantity of goods delivered			Cost of Transport goods to warehouses	Goods remaining for next period			Profit
	z_1^1	z_2^1	z_3^1	z_1^2	z_2^2	z_3^2	y_1^1	y_1^2	y_2^2	y_3^2	y_1^3
1	8	8	4	8	8	4	1960000	0	0	0	31800000
2	14	14	6	14	14	6	3310000	0	0	0	51110000
3	22	22	10	22	22	10	5270000	0	0	0	82910000
4	28	28	14	28	28	14	6860000	0	0	0	111300000
5	12	12	6	12	12	6	2940000	0	0	0	47700000

6	14	14	8	14	14	8	3550000	0	0	0	60190000
7	26	28	12	26	26	12	6460000	0	2	0	98810000
8	40	40	20	34	34	16	9800000	6	8	4	130610000
9	40	40	20	42	42	20	9800000	4	6	4	162410000
10	40	40	20	44	46	22	9800000	0	0	2	177380000
11	40	40	20	40	40	22	9800000	0	0	0	166880000
12	40	40	20	34	40	20	9800000	0	0	0	153510000
13	24	22	10	24	22	10	5430000	0	0	0	83640000
14	14	14	6	14	14	6	3310000	0	0	0	51110000
15	14	14	8	14	14	8	3550000	0	0	0	60190000
16	12	12	6	12	12	6	2940000	0	0	0	47700000
17	22	22	10	22	22	10	5270000	0	0	0	82910000
18	26	26	12	26	26	12	6250000	0	0	0	98810000
19	20	20	10	20	20	10	4900000	0	0	0	79500000
20	24	24	12	24	24	12	5880000	0	0	0	95400000
21	24	24	10	24	24	10	5640000	0	0	0	86320000
22	30	30	14	22	22	10	7230000	8	8	4	82910000
23	40	40	20	48	48	24	9800000	0	0	0	190800000
24	40	40	20	40	40	20	9800000	0	0	0	159000000

One of the crucial problems in relative to DEA models is that, there is no control over weight factors. It could be possible that each DMU could allot weights to factors in order to maximize its efficiency. In which case, there is a probability that, extremely few weights be attributed to important factors and or high weights to lower priority factors. This issue shall greatly challenge the validity of the assessment. Hence, we utilized a questionnaire that was completed by managers. The results of the completed questionnaire, demonstrates the relative importance of the inputs, intermediary and output variables and leads to imposing weight constrains according to Table 3. Likewise, the value of ε as per the opinions of managers has been taken as 0.05 in the models.

Table 3. Weights constraints in relative to the output, intermediary and input variables

Inputs		Intermediate measures	Outputs
$\frac{v_3^1}{v_2^1} \geq 1.08$	$\frac{v_3^2}{v_2^2} \geq 1.13$	$\frac{w_1^6}{w_1^5} \geq 1.16$	$\frac{u_1^3}{u_3^2} \geq 1.52$
$\frac{v_2^1}{v_1^1} \geq 1.11$	$\frac{v_2^2}{v_1^3} \geq 1.06$	$\frac{w_1^5}{w_1^3} \geq 1.2$	$\frac{u_3^2}{u_2^2} \geq 1.09$
$\frac{v_1^1}{v_5^2} \geq 1.15$	$\frac{v_1^3}{v_1^2} \geq 1.6$	$\frac{w_1^3}{w_1^2} \geq 1.05$	$\frac{u_2^2}{u_1^2} \geq 1.16$
$\frac{v_5^2}{v_4^2} \geq 1.1$		$\frac{w_1^2}{w_1^4} \geq 1.06$	$\frac{u_1^2}{u_1^1} \geq 1.28$
$\frac{v_4^2}{v_3^2} \geq 1.17$		$\frac{w_1^4}{w_1^1} \geq 1.28$	

It is necessary that the models (13), (14), (19) and 20 measure the values of $\theta_o^{1\text{-max}}$, $\theta_o^{2\text{-max}}$, $\varphi_o^{1\text{-min}}$ and $\varphi_o^{2\text{-min}}$ so as to achieve the efficiency of the cooperative approach. Moreover, model (16) renders the maximal overall efficiency from the optimistic view; and model (22) the minimal overall from the pessimistic viewpoint for the network shown in Fig. 1. The Table 4 illustrates the maximal and minimal efficiency values for the first and second stages, long with the optimal k values, which has come to hand from models (16) and (22).

Table 4. The maximum and minimum efficiency values for the first and second stages together with values for k

DMU	Optimistic View				Pessimistic View			
	$\theta_o^{1\text{-max}}$	$\theta_o^{2\text{-max}}$	k_1	k_2	$\varphi_o^{1\text{-min}}$	$\varphi_o^{2\text{-min}}$	k_1	k_2
1	1	0.74082	0	1	1	1.00000	0	0
2	1	0.84704	1	1	1	1.04438	0	1
3	1	0.92640	1	1	1	1.12473	0	3
4	1	0.96387	1	1	1	1.20415	0	2
5	1	0.83050	0	0	1	1.04083	0	0
6	1	0.87241	0	0	1	1.06920	0	1
7	1	0.96168	0	1	1	1.14529	0	5
8	1	1.00000	1	0	1	1.21250	0	2
9	1	0.81821	1	0	1	1.14786	1	3
10	1	0.86937	0	0	1	1.17132	0	54
11	1	0.98087	0	0	1	1.27914	0	24
12	1	0.99200	0	2	1	1.27670	0	19
13	1	0.92930	2	1	1	1.12489	0	3
14	1	0.84704	1	1	1	1.04438	0	1
15	1	0.87241	0	0	1	1.06920	0	1
16	1	0.83050	0	0	1	1.04083	0	0
17	1	0.92640	1	1	1	1.12473	0	3
18	1	0.94971	1	1	1	1.16522	0	5
19	1	0.91957	1	1	1	1.12249	0	0
20	1	0.94490	0	0	1	1.16332	0	1
21	1	0.93259	1	1	1	1.12677	0	3
22	1	1.00000	1	0	1	1.05155	0	0
23	1	0.78185	1	0	1	1.16244	1	0
24	1	1.00000	0	0	1	1.32665	0	2

In surveying the k values, we are aware that, in this case-study, the overall efficiency is optimized when the k values are low. This means that, the optimal efficiency values of the stages are closer to their maximum or minimum amounts. We utilized the models (1), (4) and (16) respectively, to attain the efficiency of the factory from the black box approach, the Wang generalized approach and the cooperative approach, based on the optimistic view. Then, by using Wang's approach (2009), we computed the efficiency of each of these conditions, by taking the double-frontier into account. The results of which, is given in Table (5). In Wang's approach, we considered the values of λ_1 and λ_2 by consulting experts as being equivalent to $\lambda_1 = \frac{1}{2}$ and$\lambda_2 = \frac{1}{2}$. Similarly, according to the opinions of experts, $\Delta\varepsilon$=0.01 ، M=5 was also taken under consideration.

Table 5. Comparison of the three viewpoints of efficiency: Black box, Wang and Cooperative Approaches

	Black box Approach			Wang Approach			Cooperative Approach		
DMU	$\theta_o^{overall}$	$\varphi_o^{overall}$	$\emptyset_o^{overall}$	$\theta_o^{overall}$	$\varphi_o^{overall}$	$\emptyset_o^{overall}$	$\theta_o^{overall}$	$\varphi_o^{overall}$	$\emptyset_o^{overall}$
1	0.93467	1.00000	0.96678332	0.93467	1.00000	0.96678332	0.71654	1.00789	0.84981968
2	0.96325	1.00000	0.98145300	0.96088	1.01997	0.98998422	0.80388	1.05668	0.92165281
3	0.97349	1.08166	1.02615066	0.96841	1.10469	1.03430790	0.89203	1.16104	1.01768487
4	0.97550	1.20415	1.08381194	0.97550	1.20415	1.08381194	0.93572	1.23668	1.07572589
5	0.94284	1.04083	0.99062412	0.94284	1.04083	0.99062412	0.8107	1.05781	0.92604890
6	0.97814	1.04109	1.00912426	0.97244	1.05859	1.01460103	0.87241	1.11112	0.98455685
7	0.99953	1.14576	1.07015022	0.97488	1.15490	1.06107912	0.93499	1.20292	1.06052730
8	1.00000	1.21531	1.10241099	0.98095	1.30707	1.13232959	0.97647	1.2629	1.11048816
9	0.81747	1.15514	0.97174703	0.55489	1.22452	0.82430207	0.79879	1.23201	0.99202684

10	0.86184	1.18101	1.00888139	0.59692	1.28588	0.87610929	0.84962	2.98457	1.59240395
11	0.98469	1.28830	1.12631084	0.89155	1.31147	1.08131451	0.96698	1.51914	1.21201402
12	0.97312	1.24257	1.09962253	0.94030	1.31385	1.11149141	0.95046	1.4778	1.18515390
13	0.96933	1.04258	1.00528805	0.96737	1.09446	1.02895469	0.88346	1.15489	1.01009856
14	0.96325	1.00000	0.98145300	0.96088	1.01997	0.98998422	0.80388	1.05668	0.92165281
15	0.97814	1.04109	1.00912426	0.97244	1.05859	1.01460103	0.87241	1.11112	0.98455685
16	0.94284	1.04083	0.99062412	0.94284	1.04083	0.99062412	0.8107	1.05781	0.92604890
17	0.97349	1.08166	1.02615066	0.96841	1.10469	1.03430790	0.89203	1.16104	1.01768487
18	0.97890	1.12755	1.05059920	0.97109	1.14884	1.05623247	0.91587	1.22297	1.05833904
19	0.95917	1.12249	1.03762167	0.95917	1.12249	1.03762167	0.8928	1.13679	1.00743541
20	0.96734	1.16332	1.06081382	0.96734	1.16332	1.06081382	0.92238	1.18635	1.04607146
21	0.98690	1.04187	1.01401257	0.97523	1.08719	1.02968942	0.89284	1.15677	1.01627286
22	1.00000	1.04686	1.02316176	1.00000	1.15835	1.07626669	0.97426	1.05719	1.01487828
23	0.77748	1.16969	0.95363021	0.41961	1.27620	0.731782947	0.76549	1.1834	0.95177774
24	1.00000	1.32665	1.15180293	0.94662	1.32665	1.120639738	0.98043	1.35732	1.153584521

In comparing the approaches in Table 5, it can be noted that, the optimistic approach of the black box efficiency is greater than the optimistic efficiency of Wang and the cooperative approach. Furthermore, the pessimistic efficiency of the black box approach is less than the pessimistic efficiency of the Wang pessimistic approach and that of the cooperative approach for all the DMUs. This is because the intermediary variables in the black box approach are not taken into account, but in the Wang and cooperative approach we observe a fluctuating condition. For ranking the DMUs, we utilize columns 4, 7and 10 of Table 5. Hence, the performance of 24 DMUs is rated as follows:

Table 6. Ranking of DMUs from the three viewpoints: Black box, Wang and cooperative approaches

DMU	Black box Approach	Wang Approach	Cooperative Approach
1	23	21	24
2	20	19	22
3	10	11	10
4	5	4	6
5	18	17	20
6	14	15	17
7	6	7	7
8	3	1	5
9	22	23	16
10	16	22	1
11	2	5	2
12	4	3	3
13	17	14	14
14	20	19	22
15	14	15	17
16	18	17	20
17	10	11	10
18	8	9	8
19	9	10	15
20	7	8	9
21	13	13	12
22	12	6	13
23	24	24	19
24	1	2	4

We found that for this case-study, ranking was performed by three varied approaches, but without a single outstanding response. In the following, we use the Borda technique to combine the results of the three approaches. Table 7 shows the pairwise comparison of the DMUs.

Table 7. Pairwise comparison by utilizing the Borda Method

DMU	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	-	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	M	X
2	M	-	X	X	X	X	X	X	M	X	X	X	X	O	X	X	X	X	X	X	X	X	M	X
3	M	M	-	X	M	M	X	X	M	M	X	X	M	M	M	M	O	X	X	M	M	M	M	X
4	M	M	M	-	M	M	M	X	M	M	X	X	M	M	M	M	M	M	M	M	M	M	M	X
5	M	M	X	X	-	X	X	X	M	X	X	X	X	M	X	O	X	X	X	X	X	X	M	X
6	M	M	X	X	M	-	X	X	M	M	X	X	M	M	O	M	X	X	X	X	X	X	M	X
7	M	M	M	X	M	M	-	X	M	M	X	X	M	M	M	M	M	M	M	M	M	M	M	X
8	M	M	M	M	M	M	M	-	M	M	X	M	M	M	M	M	M	M	M	M	M	M	M	X
9	M	X	X	X	X	X	X	X	-	X	X	X	X	X	X	X	X	X	X	X	X	X	M	X
10	M	M	X	X	M	X	X	X	M	-	X	X	M	M	X	M	X	X	X	X	X	X	M	X
11	M	M	M	M	M	M	M	M	M	M	-	M	M	M	M	M	M	M	M	M	M	M	M	X
12	M	M	M	M	M	M	M	X	M	M	X	-	M	M	M	M	M	M	M	M	M	M	M	X
13	M	M	X	X	M	X	X	X	M	X	X	X	-	M	X	M	X	X	X	X	X	X	M	X
14	M	O	X	X	X	X	X	X	M	X	X	X	X	-	X	X	X	X	X	X	X	X	M	X
15	M	M	X	X	M	O	X	X	M	M	X	X	M	M	-	M	X	X	X	X	X	X	M	X
16	M	M	X	X	O	X	X	X	M	X	X	X	X	M	X	-	X	X	X	X	X	X	M	X
17	M	M	O	X	M	M	X	X	M	M	X	X	M	M	M	M	-	X	X	M	M	M	M	X
18	M	M	M	X	M	M	X	X	M	M	X	X	M	M	M	M	M	-	M	X	M	M	M	X
19	M	M	M	X	M	M	X	X	M	M	X	X	M	M	M	M	M	X	-	M	M	X	M	X
20	M	M	M	X	M	M	X	X	M	M	X	X	M	M	M	M	M	M	M	-	M	M	M	X
21	M	M	X	X	M	M	X	X	M	M	X	X	M	M	M	M	X	X	X	M	-	X	M	X
22	M	M	X	X	M	M	X	X	M	M	X	X	M	M	M	M	X	X	M	M	M	-	M	X
23	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	-	X
24	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M	-

Based on scores earned by each row, the performance of 24 DMUs is rated as follows:

$$\mathrm{DMU}_{24} > \mathrm{DMU}_{11} > \mathrm{DMU}_{8} > \mathrm{DMU}_{12} > \mathrm{DMU}_{4} > \mathrm{DMU}_{7} > \mathrm{DMU}_{20} > \mathrm{DMU}_{18} > \mathrm{DMU}_{19} >$$

$$\mathrm{DMU}_{3} = \mathrm{DMU}_{17} > \mathrm{DMU}_{22} > \mathrm{DMU}_{21} > \mathrm{DMU}_{6} = \mathrm{DMU}_{15} > \mathrm{DMU}_{10} > \mathrm{DMU}_{13} > \mathrm{DMU}_{5} =$$

$$\mathrm{DMU}_{16} > \mathrm{DMU}_{2} = \mathrm{DMU}_{14} > \mathrm{DMU}_{9} > \mathrm{DMU}_{1} > \mathrm{DMU}_{23},$$

Where, the symbol " $>$ " means that the performance is better than; and the symbol "$=$ " means that the performance is equal. It should be noted that, in some cases the ranks of DMUs are identical. For example, DMU3 = DMU17. This is due to the fact that, the demand, amount of production of each commodity, amount of delivery including storage and other items during the period (3 and 17) were absolutely similar to each other and this factory had a completely uniform performance.

5 Conclusions

The black box approach neglects the internal activities of systems and evaluates performance based on the final inputs and outputs. According to the belief of many researchers, this task causes a lack of confidence in the evaluation results. By taking the internal structure of systems into account, network analysis methods cause intricacies in modeling and solving problems, such as, nonlinearity, which may result in significant changes in efficiency results. In the meanwhile, models like Wang, take the activities within the system into consideration, but in order to prevent complexities in solving models, an approach in between the black box and network is maintained. In this paper, we considered a three-stage network, in respect to the additional desirable and undesirable inputs and outputs. Thence, the efficiency of this network is viewed from the three approaches, namely, the black box approach for alleviation, the generalized Wang approach to simplify the network and the cooperative approach to compute the complexity of the network. As the cooperative approach is a complicated model, a heuristic model has been taken to solve the cooperative model in this paper. Due to the fact that, a conclusion implying only one of these two, optimistic or pessimistic views is one-sided and incomplete, so, in this paper we used the double-frontier to analyze the network. DEA application in relevance to production planning and inventory control has been observed to an extremely slight degree. In this paper, we have contemplated on an example, in the authentic world in the grounds of production planning and inventory control. In this paper, a factory producing dairy products, with a production area, warehouse premises and a delivery point, including the total costs, pertaining to production, storage, warehouse reservation, transport costs from the production area to the warehouse and from the warehouse to the delivery point, as well as the profits from sale of goods have been considered and simulated. This factory has been regarded as a dynamic network with a time period of 24 intervals.The results of these three approaches demonstrated that, the optimistic efficiency of the black box is more than the optimistic efficiency of the Wang and the cooperative approaches. It was also noted that, the pessimistic efficiency that has come to hand from the black box approach, is less than the Wang and cooperative approaches. We express the reason for this, as the lack of a presence of the intermediary variables in the black box approach. In relevance to the results of the Wang and cooperative approach, fluctuating conditions were observed. We were also aware that different approaches in respect to the data envelopment analysis do not necessarily have a similar response; and with due attention, to our lack of awareness, in respect to the correct response, we have made efforts to extract the final results by combining the information of the varied approaches. By comparing the rankings achieved from the three approaches, i.e. the black box, Wang and cooperative approaches, we utilized the Borda technique towards gaining information and in securing the final ranking units. The results of the ranking showed that, the time periods, (24) and (23) were the best and poorest respectively, in context to the efficiency within 24 phases of time. Similarly, we detected that between the time period of (1) and (24), a fluctuating condition occurred and there was an absence, of a specific system, to alleviate efficiency. The heuristic approach utilized in this paper is capable of being generalized for more complex networks. The smaller, the step size ($\Delta\varepsilon$) selected, the higher is the accuracy of the computation, though the time for solving the problem increases. Hence, the step size ($\Delta\varepsilon$) which specifies the accuracy of resolving the problem and the time taken for this, should be considered by the managers. We have put this research at the disposal of the managers, so that the best decisions can be adopted for the abovementioned factory. For researches in the future, modeling with imprecise and random data is suggested.

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