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A novel radial super-efficiency DEA model handling negative data

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Abstract

Super-efficiency model in the presence of negative data is a relatively neglected issue in the DEA field. The existing super-efficiency models have some shortcomings in practice. In this paper, a novel VRS radial super-efficiency DEA model based on Directional Distance Function (DDF) is proposed to provide a complete ranking order of units (including efficient and inefficient ones). The proposed model is feasible no matter whether data are non-negative or not. This model shows more reliability on differentiating efficient units from inefficient ones via a new bounded super-efficiency measure. It can project each unit onto the super-efficiency frontier along a new non-negative direction and produce improved targets for inefficient units. The model overcomes the infeasibility issues occur in Nerlove–Luenberger supper-efficiency model. The proposed model conveys good properties such as monotonicity, unit invariance and translation invariance. Apart from numerical examples, an empirical study in bank sector demonstrates the superiority of the proposed model.

Keywords: Data envelopment analysis; Super-efficiency model; Negative data; DDF model.

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1. Introduction

Data Envelopment Analysis (DEA) is a powerful tool in the context of production management for performance measurement. The purpose of DEA is to measure the relative efficiency of a set of decision making units (DMUs) where multiple inputs convert into multiple outputs (Charnes et al., 1978). Conventional DEA models assume non-negative values for inputs and outputs. However, there are many applications in which one or more inputs and/or outputs are necessarily negative such as the performance analysis of socially responsible and mutual funds (Basso and Funari, 2014), and the macroeconomic performance where "rate of growth of GDP per capita" can be either negative or positive (Lovell, 1995). In DEA literature, there have been various approaches for dealing with unrestricted in sign variables.

Lovell and Pastor (1995) and Pastor (1996) approached negative data using a translation invariance classification, for the first time. They added a number to all the data to convert them to positive ones. Many DEA models such as CCR do not have translation invariance property to apply the treatment of negative data (Ali and Seiford, 1990). Many researches have been carried out in the DEA literature to address the occurrence of the negative data.

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Silva Portela et al. (2004) proposed range directional measure (RDM) model using some variations of the DDF. Sharp et al. (2007) extended a modified slack-based measure for negative data, inspired by the Silva's RDM model. Emrouznejad et al. (2010) proposed a Semi-Oriented Radial Measure (SORM). While Kerstens and Van de Woestyne (2011) modified the traditional proportional distance function, Cheng et al. (2013) suggested variant of the traditional input- or output-oriented radial efficiency measure to handle negative inputs and outputs. Kerstens and Van de Woestyne (2014) highlighted some shortcomings in Cheng's method using a more general case of the DDF proposed by Kerstens and Van de Woestyne (2011). An overview of the large number of DEA modeling approaches can be found in Pastor and Ruiz (2007), and Pastor and Aparicio (2015). All the above presented models did not study ranking units in the presence of negative data. As known, in the absence of negative data, the classical super-efficiency model under constant returns to scale (CRS) does not suffer from the infeasibility problem, but the super-efficiency model based upon the variable returns to scale (VRS) model of Banker et al. (1984) may be infeasible for a given unit under evaluation (see, e.g., Seiford and Zhu (1999), Chen and Liang (2011), Lee et al. (2011), Lee and Zhu (2012)). Many modified VRS radial super-efficiency DEA models (see, e.g., Chen (2005), Cook et al. (2009), Lee et al. (2011)) were proposed to address the infeasibility issue. On the other hand, Ray (2008) suggested the VRS Nerlove-Luenberger super-efficiency DEA model, based on the DDF model and showed that apart from two exceptions the model is feasible. By choosing proper directions, Chen et al. (2013) proposed a DDF-based VRS super-efficiency DEA model to address the infeasibility issues mentioned in Ray (2008). Lin and Chen (2015) consider the model in Chen et al. (2013) when zero data exist in outputs. All these modified super-efficiency DEA models are proposed for the non-negative data and the infeasibility issue when there are negative inputs or outputs still exists. For the first time, Hadi-Vencheh and Esmaeilzadeh (2013) provided a super-efficiency model (VE model) based on the RDM model for ranking units in the presence of negative data. However, Pourmahmoud et al. (2016) highlighted some shortcomings in VE model and proved that their model suffers from the common infeasibility and unboundedness problems. Pourmahmoud et al. (2016) showed that the VE model will be always feasible when all range of possible improvements are strictly positive. In addition, they defined four cases in which the envelopment form of the VE model is infeasible. In general, the infeasibility occurs when (i) there exists zero range of possible improvements in inputs and/or outputs of the evaluated DMU and (ii) the corresponding inputs (outputs) with a zero amount of improvement of the DMU under evaluation are outside of the production possibility set (PPS) spanned by the inputs (outputs) of the remaining DMUs. Recently, Lin and Chen (2017) proposed a novel DDF-based VRS radial super-efficiency DEA model which is feasible and is able to handle negative data. This paper highlights some cases that their model is not responding for ranking of all units for example when units consume the same inputs. Apart from Hadi-Vencheh and Esmaeilzadeh (2013) and Lin and Chen (2017), super-efficiency models with negative data have received no attention in the literature. The contribution of this paper is seven fold:

1. A novel DDF-based VRS super-efficiency model interacting with negative data is proposed.
2. By choosing a new non-negative improvement direction, a novel DDF-based DEA model is introduced.
3. The proposed model is always feasible and conveys good properties such as unit invariance, translation invariance and monotonicity.
4. This study shows that in distinguishing units, proposed model shows higher reliability than the other super-efficiency models compared in this study, due to introducing a bounded super-efficiency measure.
5. The model produces improved targets for inefficient units.

6. The infeasibility issues mentioned in Ray (2008) does not occur under our proposed model.
7. Numerical examples and an empirical study in bank sector demonstrate the applicability and the superiority of the proposed model.

The rest of the paper is outlined as follows. Section 2 briefly presents the concept of DDF model, DDF-based super-efficiency model and the model proposed by Lin and Chen (2017). In Section 3, a novel radial DDF-based super-efficiency model handling negative data is introduced. In section 4, the proposed model is applied to two numerical examples.
The penultimate section is devoted to an illustration application and finally Section 6 concludes this study.

2. Preliminaries

2.1. DDF model

Consider a set of n observed DMUs, $\{DMU_j \ (j = 1,2,\dots,n)\}$ where each observation transforms m inputs, x_{ij} $(i = 1,2,\dots,m)$, into outputs, y_{rj} $(r = 1,2,\dots,s)$. Consider an input-output bundle for $DMU_o(x_o, y_o)$ and a reference input-output bundle (g^x, g^y). Furthermore, assume that all data are non-negative. Production possibility set $\mathrm{T_o(x,y)}$ from the observed input-output for n DMUs can be defined as follows:

$$T_o(x,y) = \left\{(x,y): x \geq \sum_{j=1}^{n} \lambda_j x_j;\ y \leq \sum_{j=1}^{n} \lambda_j y_j; \sum_{j=1}^{n} \lambda_j = 1\, ; \lambda_j \geq 0; (j = 1,2,\dots,n)\right\}$$

which is constructed assuming convexity, free disposibility of inputs and outputs, and VRS. Based on T_o, the DDF regarding $T_o(x,y)$can be expressed as follows (Chambers et al., 1996):

$$D(x_o, y_o; g^x, g^y) = \max \beta : (x_o - g^x, y_o + g^y) \in T_o. \qquad (1)$$

The reference bundle (g^x, g^y) can be chosen in an arbitrary way and this makes the DDF varies with reference to the evaluated unit. The VRS DEA formulation for model (1) is as follows:

$$\begin{aligned}
&max \quad \beta \\
&s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} - \beta g^x, \quad \forall i, \qquad (2)\\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} + \beta g^y, \quad \forall r,\\
&\qquad \sum_{j=1}^{n} \lambda_j = 1,\\
&\qquad \lambda_j \geq 0,\\
&\qquad \beta \ free.
\end{aligned}$$

Model (2) combines the features of both an input- and output-oriented models in which each input and output of the unit under assessment are decreased and increased respectively, at the same time by the same portion β.The factor β^*as the optimal value of β in model (2) is the Nerlove–Luenberger (N–L) measure of technical inefficiency for the evaluated unit. By implication, its efficiency equals $1 - \beta^*$ (Ray (2008)).

2.2. Super-efficiency model based on DDF

The super-efficiency version of model (1) is obtained when DMU_o under evaluation is removed from the reference set. $T_o^s(x,y)$ of super-efficiency for n DMUs can be defined as follows:

$$T_o^s(x,y) = \left\{(x,y): x \geq \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_j;\ y \leq \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_j; \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j = 1\, ; \lambda_j \geq 0; (j = 1,2,\dots,n; j \neq o)\right\}$$

The super-efficiency based on DDF model (1) is as follows:

$$D(x_o, y_o; g^x, g^y) = \max \beta : (x_o - g^x, y_o + g^y) \in T_o^s.$$

DDF-based super-efficiency DEA model can be established as follows:

$$\begin{aligned} \max \quad & \beta \\ \text{s.t.} \quad & \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j x_{ij} \leq x_{io} - \beta g^x, \quad \forall i, \\ & \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j y_{rj} \geq y_{ro} + \beta g^y, \quad \forall r, \\ & \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0, \ \forall j, j \neq o \\ & \beta \ \text{free} . \end{aligned} \tag{3}$$

Ray (2008) defined the super-efficiency score of the evaluated DMU_o equals $1-\beta_o^*$, where β_o^* is the optimum value of model (3). The smaller the value of β_o^*, the more efficient the DMU_o is. For any efficient DMU_o, $1-\beta_o^*$ is no less than 1.

The direction vector (g^x, g^y) should be non-negative and non-zero, and can be chosen in an arbitrary way (Chen *et al.*(2013), Ray (2008)). Briec and Kerstens (2009) indicated that model (3) cannot guarantee the feasibility, if the direction is a constant vector and the output direction vector is non-zero. Hence, g^x and $g^y$are often considered as the function of $x_o$and y_o. If all input and output data are non-negative, the standard DDF for the $DMU_o$is adopted by choosing (x_o, y_o) as (g^x, g^y) (Chambers *et al*. (1998), Briec (1997)) and the N-L super-efficiency model (NLS model) is obtained. The NLS model is very often feasible for non-negative data, but it fails in the two cases (Ray (2008)). To address these infeasibility issues, Chen *et al.* (2013) selected a new reference input–output bundle for the DDF and propose a modified DDF-based VRS super-efficiency model. However Lin and Chen (2015) showed that the model proposed by Chen *et al.* (2013) does not fully eliminate the infeasibility issue in Ray (2008). In this regards, Lin and Chen (2015) proposed a modified DDF-based super-efficiency DEA model (LCS model) by choosing $\left(x_{io} + \max_{j\neq o}\{x_{ij}\}, y_{ro}\right)$ as (g^x, g^y). The LCS model successfully addresses the infeasibility issue in conventional VRS radial super-efficiency DEA models and the NLS model under non-negative data.

2.3. Proposed model by Lin and Chen (2017)

Lin and Chen (2017) showed that in the presence of negative data, both the NLS and LCS models might be infeasible. This is because their related direction vectors, (x_{io}, y_{ro}) and $\left(x_{io} + \max_{j\neq o}\{x_{ij}\}, y_{ro}\right)$, might be negative, which could result in the $DMU_o$to be further away from the super-efficiency frontier and thus lead to infeasibility. Accordingly, they choose a new direction vector which is always non-negative and non-zero, independent of inputs and outputs being non-negative or not. Their proposed model is as follows:

$$\begin{aligned} \max \quad & \beta \\ \text{s.t.} \quad & \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j x_{ij} \leq (1-\beta) x_{io} - a_i \beta, \quad \forall i, \\ & \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j y_{rj} \geq (1+\beta) y_{ro} - b_r \beta, \quad \forall r, \\ & \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0, \ \forall j, j \neq o \\ & \beta \ \text{free} \end{aligned} \tag{4}$$

Where $a_i = k * \max_{j=1,2,\dots,n}\{|x_{ij}|\}$, $i = 1,2,\dots,m$ and $b_r = \min_{j=1,2,\dots,n}\{y_{rj}\}$, $r = 1,2,\dots,s$; k is a constant, satisfying $k \geq 3$.

Consider the numerical example presented in Table 1 where there are eight DMUs with one positive input(x), and two free in sign-valued outputs $(y_1 \ and \ y_2)$.

Table 1. Numerical example

DMUs	x	y_1	y_2
A	1	-6	5
B	1	-6	3
C	1	-5	-2
D	1	-2	-5
E	1	2	-6
F	1	-3.5	3.5
G	1	6.5	-3
H	1	5	2

The results of applying model (4) to the units in Table 1 are presented in Table 2. The optimal values of $1-\beta^*$ besides the optimal slack values ($s^*;t_1^*,t_2^*$) are shown in columns two-five. The input and outputs projections $(x^*; y_1^*, y_2^*)$ are represented in the columns six-eight. Projection points are computed by inserting the optimal value in the right-hand side of the input and output inequalities in model (4).

Table 2. The results of numerical example

DMUs	$1-\beta^*$	s^*	t_1^*	t_2^*	x^*	y_1^*	y_2^*
A	1.1364	0.5455	2.5000	0.0000	1.5455	-6.0000	3.5000
B	1.0000	0.0000	7.3333	0.0000	1.0000	-6.0000	3.0000
C	1.0000	0.0000	10.0000	4.0000	1.0000	-5.0000	-2.0000
D	1.0000	0.0000	7.0000	7.0000	1.0000	-2.0000	-5.0000
E	1.0000	0.0000	3.0000	8.0000	1.0000	2.0000	-6.0000
F	1.0000	0.0000	3.0000	0.0000	1.0000	-3.5000	3.5000
G	1.1200	0.4800	0.0000	5.3600	1.4800	5.0000	-3.3600
H	1.2657	1.0627	0.0000	0.0000	2.0627	2.0776	-0.1254

Table 2 reports that $\beta_B^* = \beta_C^* = \beta_D^* = \beta_E^* = \beta_F^* = 0$, $\beta_A^* = -0.1364$, $\beta_G^* = -0.1200$ and $\beta_H^* = -0.2657$. DMUs A, G and H are Pareto-efficient, while DMUs B, C, D, E and F are inefficient due to the optimal slack-values. Table 1 shows that all the units are on the frontier in their input components meaning that input level is efficient; but due to illogical results for DMUs A, G and H the input projections are not on the efficient frontier, as represented in Table 2. This is because $x_{io} + a_i > 0,\ \forall i$ for each $o \in \{1, 2, \dots, n\}$ and model (4) uses a unified changing rate β for both inputs and outputs. Thus, when units consume the same inputs, our expectation is $\beta^* = 0$ and $x^* = 1$ for all units whether efficient or inefficient. This demonstrates that the optimal values of $\beta^*$and the projection points for DMUs A, G and H are illogical results. Consequently, using the $1-\beta^*$ as the super-efficiency measure, model (4) is unable to provide a complete ranking order for all units. Note that this expectation is not true, when units produce the same outputs; because in this case $y_{ro} - b_r = 0,\ \forall r$ for each $o \in \{1, 2, \dots, n\}$ and the output constraints in model (4) is disappeared due to convexity constraint. Thus, the optimal value of $\beta^*$for each DMU is changed and the output projection for all units equals one.

3. Proposed super-efficiency model

In this section, a new super-efficiency DEA model based on DDF is proposed for ranking all DMUs in the presence of negative data. The proposed DDF-based super-efficiency model by choosing $\left(x_i^{max} - x_{io}, y_{ro} - y_r^{min}\right)$ as the new reference input-output bundle is as follows:

$$
\begin{aligned}
\max \quad & \delta \\
s.t. \quad & \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j x_{ij} \le x_{io} - (x_i^{max} - x_{io})\delta, \quad \forall i, \\
& \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j y_{rj} \ge y_{ro} + (y_{ro} - y_r^{min})\delta, \quad \forall r, \\
& \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j = 1, \\
& \lambda_j \ge 0, \ \forall j, \ j \ne o \\
& \delta \ \ \text{free}
\end{aligned}
\tag{5}
$$

where

$$x_i^{max} = \max_{j=1,2,\dots,n} \{x_{ij}\}, \ i = 1,2,\dots,m$$

and

$$y_r^{min} = \min_{j=1,2,\dots,n} \{y_{rj}\}, \ r = 1,2,\dots,s.$$

Note that the new direction is non-negative[1].

It is proved that for each unit the optimal value of model (5) is $\delta^* \ge 0$ for $(x_{io}, y_{ro}) \in T_o^s$ and $\delta^* \le 0$ for $(x_{io}, y_{ro}) \notin T_o^s$. To have a ranking order for all units, a measure is needed which is bigger than 1 for efficient DMUs and also between 0 and 1 for inefficient ones. In so doing, $\rho^* = 1 - \frac{\delta^*}{\hat{\delta}_o}$ is considered as a new measure of super-efficiency where $\hat{\delta}_o$ is defined as:

$$
\hat{\delta}_o = 1 + \min \left\{ \min_i \left(\frac{x_{io} - x_i^{min}}{x_i^{max} - x_{io}}; \begin{matrix} x_i^{max} - x_{io} \ne 0, \\ x_{io} - x_i^{min} \ne 0 \end{matrix} \right), \min_r \left(\frac{y_r^{max} - y_{ro}}{y_{ro} - y_r^{min}}; \begin{matrix} y_r^{max} - y_{ro} \ne 0, \\ y_{ro} - y_r^{min} \ne 0 \end{matrix} \right) \right\}
\tag{6}
$$

where

$$x_i^{min} = \min_{j=1,2,\dots,n} \{x_{ij}\}, \ i = 1,2,\dots,m$$

and

$$y_r^{max} = \max_{j=1,2,\dots,n} \{y_{rj}\}, \ r = 1,2,\dots,s.$$

Where numerator and/or denominator are zero, the corresponding term is dropped.

Theorem 1. Model (5) is always feasible and the following inequalities are hold:

a) $0 < \rho^* \le 1$ for $(x_{io}, y_{ro}) \in T_o^s$;
b) $1 < \rho^* \le 2$ for $(x_{io}, y_{ro}) \notin T_o^s$.

Proof. The proof is given in Appendix A,

Corollary 1. $\rho^* \in (0,2]$ for all units.

From model (5) the output-projections for DMU_o are

$$y_{ro}^* = y_{ro} + \left(y_{ro} - y_r^{min}\right)\delta^*, \qquad \forall r$$

where $\delta^*$is the optimal value of model (5). According to the results mentioned in Appendix A,

$$y_{ro}^* = y_{ro} + \left(y_{ro} - y_r^{min}\right)\delta^* \ge y_{ro}, \text{ when } (x_{io}, y_{ro}) \in T_o^s$$

[1] The direction can be zero when $x_i^{max} = x_{io}, \forall i$ and $y_r^{min} = y_{ro}, \ \forall r$. However, in this case the evaluated unit is absolutely the worst inefficient one.

and

$$y_{ro}^* = y_{ro} + (y_{ro} - y_r^{min})\delta^* \geq y_{ro} - (y_{ro} - y_r^{min}) = y_r^{min}, \text{ when } (x_{io}, y_{ro}) \notin T_o^s.$$

Therefore, the following Lemma is hold.

Lemma 1. For the data set with non-negative outputs, $y_{ro}^* \geq 0$ satisfies for any DMU_o $(o \in \{1,2,\dots,n\})$.

Corollary 2. From Theorem 1 and Lemma 1, it is concluded that the infeasibility issues occur in NLS model does not occur under our proposed model.

Theorem 2. Model (5) is unit invariant (The proof is given in Appendix B).

Theorem 3. Model (5) is translation invariant (The proof is given in Appendix C).

Theorem 4. If inputs (outputs) of the DMU_o are reduced (increased), the optimal value of model (5) does not increase (The proof is given in Appendix D).

Further examination of the proposed method is made by applying DMUs in Table 1. Table 3 reports the results when proposed model is applied to the numerical example in Table 1. The optimal solutions of the proposed model δ^* besides $\hat{\delta}_o$ are shown in the second and third columns of Table 3, respectively; and the super-efficiency measure ρ^* is presented in the fourth column. The columns five-seven of Table 3 show the projection point for a unit under evaluation.

Table 3. The results of applying proposed model for data set in Table 1

DMUs	δ^*	$\hat{\delta}_o$	ρ^*	x^*	y_1^*	y_2^*	*Ranking order*
A	-0.1364	1.0000	1.1364	1.0000	-6.0000	3.5000	2
B	0.2222	1.2222	0.8182	1.0000	-6.0000	5.0000	5
C	1.5745	2.7500	0.4275	1.0000	-3.4255	4.2979	7
D	2.1163	3.1250	0.3228	1.0000	6.4651	-2.8837	8
E	0.5625	1.5625	0.6400	1.0000	6.5000	-6.0000	6
F	0.0804	1.1579	0.9306	1.0000	-3.2991	4.2634	4
G	-0.1200	3.6667	1.0327	1.0000	5.0000	-3.3600	3
H	-0.2657	1.1364	1.2338	1.0000	2.0776	-0.1254	1

The results show that DMUs A, G and H are efficient; since their supper-efficiency measures are greater than one. However, units B, C, D, E and F are inefficient, since their supper-efficiency measures are less than one. As seen, the proposed model provides improved targets for inefficient units. Column five shows that $x^* = 1$ for all units, and this logical outcome was expected. The proposed model provides ranking order for all units, shown in column eight:

$$H \succ A \succ G \succ F \succ B \succ E \succ C \succ D.$$

4. Numerical example

In this section, two numerical examples are used to show the applicability and merits of the proposed model.

Example 1.

Consider the data set of "the notional effluent processing system" from Sharp *et al.* (2007) presented in Table 4. There are 13 DMUs, with two inputs $\{x_1, x_2\}$ and three outputs $\{y_1, y_2, y_3\}$: one positive input (cost), one non-positive input (effluent), one positive output (saleable output), and two non-positive outputs (methane and CO2).

Table 4. Data sets used in Example 1, extracted from Sharp

DMUs	x_1	x_2	y_1	y_2	y_3
A	1.03	-0.05	0.56	-0.09	-0.44
B	1.75	-0.17	0.74	-0.24	-0.31
C	1.44	-0.56	1.37	-0.35	-0.21
D	10.8	-0.22	5.61	-0.98	-3.79
E	1.30	-0.07	0.49	-1.08	-0.34
F	1.98	-0.10	1.61	-0.44	-0.34
G	0.97	-0.17	0.82	-0.08	-0.43
H	9.82	-2.32	5.61	-1.42	-1.94
I	1.59	0.00	0.52	0.00	-0.37
J	5.96	-0.15	2.14	-0.52	-0.18
K	1.29	-0.11	0.57	0.00	-0.24
L	2.38	-0.25	0.57	-0.67	-0.43
M	10.30	-0.16	9.56	-0.58	0.00

Table 5, shows the results of applying model (4) and model (5) on data sets used in Table 4. The second column represents the super-efficiency provided by model (4) and the columns four-six show the results obtained after model (5) is applied. As seen in the second and the sixth columns in Table 5, both models are feasible for all units and they can differentiate the performance of both efficient and inefficient units for used data set. The optimal values of δ^*, $\hat{\delta}_o$ and the super-efficiency measure ρ^* are represented in the columns four-six in Table 5, respectively. For both models, DMUs C, G, H, K and M are efficient, since their supper-efficiency measures are greater than 1 and the others are inefficient, since their supper-efficiency measures are less than 1. Columns three and seven which represents the ranking order of all units using model (4) and model (5), respectively, reflect that the ranking orders of both models are close; however their super-efficiency measures are different. This is due to the different improvement directions and different measures.

Table 5. Applying the proposed model for data set in Table 4

DMUs	$1-\beta^*$	*Ranking order*	δ^*	$\hat{\delta}_o$	ρ^*	*Ranking order*
A	0.9982	7	0.0050	1.0061	0.9950	7
B	0.9863	10	0.0261	1.0862	0.9760	10
C	1.0412	3	-0.1126	1.0502	1.1072	3
D	0.9192	13	0.6954	1.7715	0.6074	13
E	0.9955	8	0.0117	1.0347	0.9887	8
F	0.9921	9	0.0279	1.0986	0.9746	11
G	1.0108	5	-0.0307	1.0597	1.0290	5
H	1.4023	2	-0.7654	1.7715	1.4321	1
I	1.0000	6	0.0000	1.0673	1.0000	6
J	0.9829	11	0.0207	1.0499	0.9803	9
K	1.0292	4	-0.0396	1.0336	1.0383	4
L	0.9694	12	0.0681	1.1280	0.9397	12
M	1.5402	1	-0.5402	1.6905	1.3195	2

As can be seen from Table 5, all the super-efficiency scores yielded by model (5) for inefficient units are less than or equal to those generated by model (4). The super-efficiency scores vary from 0.9192 to 1.5402 under the model (4), whereas they vary from 0.6074 to 1.4321 under our proposed model. Obviously, the super-efficiency scores yielded from model (5) have bigger changing ranges for units in comparison with model (4). This shows the merits of the proposed model. From Table 5, DMUs M and D have the best and the worst performance under model (4), respectively, whereas DMUs H and D have the best and the worst performance under proposed model (5), respectively. Table 6 shows the target input-output values of inefficient units, determined by model (5).

Table 6. Improved targets for inefficient DMUs provided by model (5)

DMUs	x_1^*	x_2^*	y_1^*	y_2^*	y_3^*
A	0.9813	-0.0502	0.5603	-0.0834	-0.4233
B	1.5138	-0.1744	0.7465	-0.2092	-0.2192
D	10.8000	-0.3730	9.1705	-0.6740	-3.7900
E	1.1893	-0.0708	0.4900	-1.0760	-0.2998
F	1.7335	-0.1028	1.6413	-0.4126	-0.2436
I	1.5900	0.0000	0.5200	0.0000	-0.3700
J	5.8598	-0.1531	2.1742	-0.5014	-0.1052
L	1.8069	-0.2670	0.5754	-0.6190	-0.2013

Lin and Chen (2017) calculated the improved targets for inefficient units. The proposed model demonstrates that in each inefficient unit, the inputs and the outputs should be reduced and expanded, respectively, in order to tend to the super-efficiency frontier. Hence, the proposed model the same as Lin and Chen's model can provide improved target inputs and outputs for all the inefficient units but there are variations due to having different directions in their movements to reach the super-efficiency frontier.

Example 2

In this example, 7 hypothetical DMUs with two inputs $\{x_1, x_2\}$ and one output $\{y\}$, are assumed as listed in Table 7. Note that one input (x_1) and the output (y) contain negative values for some units.

Table 7. Assumed data sets for Example 2

DMUs	x_1	x_2	y
A	-3	10	2.5
B	-2	8	4
C	2.5	5	-0.1
D	-5	1	3.2
E	4.5	6	2
F	-4	5.5	4.5
G	2	9	-1

The outcomes after applying model (4) and model (5) to the assumed data set are reported in Table 8. The second column represents the super-efficiency provided by model (4) and the columns four-six show the results obtained after proposed model is applied. Both models are able to provide the feasible solutions for all units and obtain the super-efficiency measures for them, as shown in the second and sixth columns. From the results it can be concluded that in both models DMUs D and F are efficient, since their supper-efficiency measures are greater than 1. All units other than DMUs D and F are inefficient, since their supper-efficiency measures are less than 1. The outcomes for the ranking orders using model (4) and model (5) shown in the third and seventh columns, respectively represents that DMU D is superior to other units in both models. Their ranking orders are different however.

Table 8. Outcomes after applying the assume data on the model (4) and the model (5)

DMUs	$1-\beta^*$	Ranking order	δ^*	$\hat{\delta}_o$	ρ^*	Ranking Order
A	0.8333	6	0.2491	1.2667	0.8034	4
B	0.9235	3	0.1000	1.1000	0.9091	3
C	0.8857	4	0.8000	1.8000	0.5556	6
D	1.1441	1	-0.4704	1.3095	1.3592	1
E	0.8611	5	0.6364	1.8333	0.6529	5
F	1.1225	2	-0.1330	1.1176	1.1190	2
G	0.7949	7	2.8000	3.8000	0.2632	7

Table 9 represents the improved targets for inefficient units obtained from proposed model.

Table 9. Improved targets for inefficient units provided by model (5)

DMUs	x_1^*	x_2^*	y^*
A	-4.8679	10.0000	3.3717
B	-2.6500	7.8000	4.5000
C	0.9000	1.0000	0.6200
E	4.5000	3.4545	3.9091
G	-5.0000	6.2000	-1.0000

As can be seen from the above two examples, all the super-efficiency scores yielded by model (5) for inefficient units are less than or equal to those generated by model (4). Thus, the results show more reliability and responsibility of the proposed model. From the theoretical analyses it is concluded that, the same as Lin and Chen's model, the proposed model can deal with the data set with free in sign values and can provide improved targets for inefficient units.

To examine the monotonicity of the proposed model, consider the data set used in Table 7. Suppose that the first input of DMU F is decreased from -4 to -104 and its output is increased from 4.5 to 204.5 in the following way: x_1=-4-L, and y=4.5+2*L, where L increases from 0 to 100 with the step size equals to 1. When L increases from 0 to 100, the optimal value of model (5) i.e., δ^* decreases gradually from -0.1330 to -0.9757. Figure 1 shows the changes of δ^* with respect to L for DMU F. As can be seen, the value of δ^* monotonically decreases with the increase of L and this confirms Theorem 4 which claims the monotonicity of the proposed model.

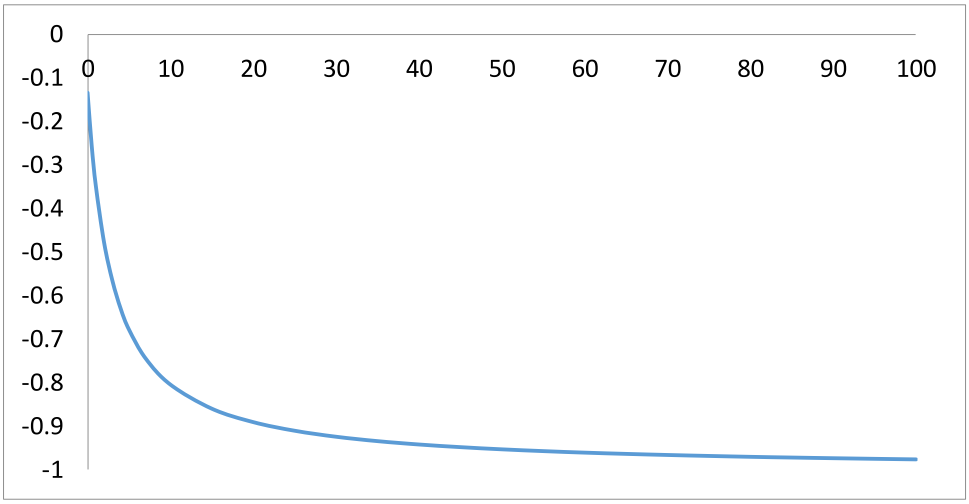

Figure 1. The change of the optimal value of model (5) for DMU F

The next section provides a numerical illustration to show the superiority and flexibility of the proposed model in comparison with Lin and Chen's proposed model and VE model.

5. An empirical application

In this section a real world data of the 61 banks in the GCC[1] countries is used to show the applicability and merits of the proposed model (5) in comparison with VE model, and Lin and Chen's proposed model (4). In this evaluation, the input variables are total assets, capital and deposits. The output variables are loans and equity in each branch. Note that the last output could take both positive and negative values among the banks. For full definitions of variables see Emrouznejad and Anouze (2010). Table 10 below shows the descriptive statistics of the variables.

Table 10. Descriptive statistics of the banks data

Variables (million $)	Min	Max	Mean	Median	St. Dev
Inputs					
Assets	252.49	29313	5569.16	2390.31	6667.20
Equity	50.19	2381.04	627.15	398.84	615.02
Deposit	26.05	25251.31	4495.24	2006.6	5560.15
Outputs					
Loan	120.97	15379	2777.32	1427.89	3222.04
Profit	-51	647.7	93.11	41.59	128.45

The outcomes are reported in Table 11.

Table 11. Outcomes after applying the assume data on three models: VE model, Lin and Chen's model and the proposed model

Banks	VE model	$1-\beta^*$	Ranking Order	δ^*	$\hat{\delta}_o$	ρ^*	Ranking Order
1	***Infeasible***	1.0052	6	-0.0157	13.3825	1.0012	13
2	0.9639	0.9904	37	0.0323	1.3040	0.9752	43
3	1.0077	1.0016	11	-0.0049	1.0583	1.0046	10
4	0.9327	0.9796	46	0.0674	1.1656	0.9421	55
5	1.0121	1.0030	8	-0.0089	1.0083	1.0088	7
6	0.9038	0.9743	49	0.0890	1.1770	0.9244	57
7	1.523	1.0946	3	-0.2452	1.2046	1.2036	1
8	0.9442	0.9869	41	0.0422	1.0478	0.9597	51
9	***Infeasible***	1.1699	2	-0.1699	3.7898	1.0448	3
10	0.9009	0.9673	52	0.1298	1.6258	0.9202	58

[1]The Gulf Cooperation Council (GCC), is a trade bloc involving the six Arab states of the Persian Gulf with many economic and social objectives (for full details see www.gcc-sg.org).

11	0.997	0.9994	19	0.0018	1.0106	0.9983	23
12	0.9879	0.9977	27	0.0071	1.0102	0.9929	32
13	0.9677	0.9910	36	0.0284	1.0456	0.9728	46
14	0.9991	0.9998	16	0.0006	1.0226	0.9994	17
15	0.9920	0.9983	25	0.0054	1.1486	0.9953	30
16	0.9602	0.9873	40	0.0402	1.1142	0.9639	47
17	0.9014	0.9752	48	0.0875	1.1825	0.9260	56
18	0.9951	0.9989	22	0.0033	1.0123	0.9967	26
19	0.975	0.9936	33	0.0235	1.6556	0.9858	36
20	0.8804	0.9689	50	0.1142	1.2379	0.9078	60
21	0.9969	0.9992	21	0.0023	1.0189	0.9977	25
22	1.0051	1.0015	12	-0.0043	1.0705	1.0040	11
23	0.9899	0.9971	28	0.0093	1.1986	0.9923	33
24	0.9952	0.9985	23	0.0045	1.0519	0.9957	27
25	0.973	0.9916	35	0.0261	1.0654	0.9755	42
26	1.0585	1.0130	4	-0.0422	1.1938	1.0353	4
27	0.9962	0.9993	20	0.0022	1.2542	0.9982	24
28	0.9802	0.9946	31	0.0170	1.0395	0.9836	39
29	0.9825	0.9952	30	0.0159	1.2575	0.9873	35
30	0.7877	0.9194	53	0.3915	1.7512	0.7765	61
31	1.0004	1.0001	15	-0.0003	1.0062	1.0003	16
32	0.9986	0.9996	18	0.0012	1.0174	0.9988	21
33	0.9991	0.9997	17	0.0009	1.3407	0.9993	18
34	0.9754	0.9946	31	0.017	1.0580	0.9840	38
35	0.9643	0.9901	38	0.0325	1.2605	0.9742	45
36	0.9984	0.9996	18	0.0012	1.0102	0.9989	20
37	0.9489	0.9850	43	0.0582	1.5556	0.9626	49
38	0.9563	0.9867	42	0.0421	1.1018	0.9618	50
39	0.8865	0.9675	51	0.1207	1.4626	0.9174	59
40	0.9949	0.9985	23	0.0045	1.0168	0.9956	28
41	0.9179	0.9794	47	0.0872	1.7237	0.9494	54
42	0.9837	0.9967	29	0.0105	1.0794	0.9903	34
43	0.9935	0.9982	26	0.0054	1.0191	0.9947	31
44	0.9979	0.9996	18	0.0013	1.0140	0.9987	22

45	1.0017	1.0003	14	-0.0007	1.0071	1.0007	15
46	***Infeasible***	1.2310	1	-0.231	1.3627	1.1695	2
47	0.994	0.9984	24	0.0048	1.0243	0.9953	29
48	1.0249	1.0036	7	-0.0107	1.0001	1.0107	6
49	1.0025	1.0008	13	-0.0023	1.0489	1.0022	12
50	1.0206	1.0028	9	-0.0087	1.0070	1.0087	8
51	1.0059	1.0020	10	-0.006	1.0290	1.0058	9
52	0.9301	0.9823	44	0.0655	1.3850	0.9527	52
53	0.9157	0.9799	45	0.084	1.6664	0.9496	53
54	1.028	1.0058	5	-0.0231	1.3004	1.0177	5
55	0.9562	0.9884	39	0.0441	1.7124	0.9742	44
56	0.9794	0.9939	32	0.0206	1.3131	0.9843	37
57	0.9560	0.9867	42	0.0444	1.2116	0.9634	48
58	0.9938	0.9997	17	0.0008	1.0009	0.9992	19
59	0.9755	0.9926	34	0.0238	1.1484	0.9793	41
60	1.0009	1.0003	14	-0.0008	1.0112	1.0008	14
61	0.9681	0.9939	32	0.0189	1.0190	0.9814	40

As it is shown in the second column in Table 11, VE model is infeasible for DMUs 1, 9 and 46. Both model (4) and model (5) are feasible for all units; however their super-efficiency measures are different as represented in the third and seventh columns. As can be seen, all the super-efficiency scores yielded by model (5) for inefficient units are less than or equal to those generated by model (4) as shown in Figure 2. The super-efficiency scores vary from 0.9194 to 1.2310 under the Lin and Chen's model, whereas they vary from 0.7765 to 1.2310 under our proposed model. From Figure 2, in general, the super-efficiency scores obtained from model (4) is around 1.0000 for inefficient DMUs, whereas the scores yielded from model (5) have bigger changing ranges for inefficient ones. From Table 11, DMUs 46 and 30 have the best and the worst performance, respectively under both models. Column seven presents a complete ranking order for all units (both efficient and inefficient ones) using proposed model. However, from column four, Lin and Chen's model cannot put discriminations between some inefficient units: between DMUs 45 and 60, DMUs 33 and 58, DMUs 32, 36 and 44, DMUs 24 and 40, DMUs 28 and 34, DMUs 56 and 61, and also DMUs 38 and 57. This shows that the proposed model is more responsive than model (4) and it can differentiate the units better than model (4).

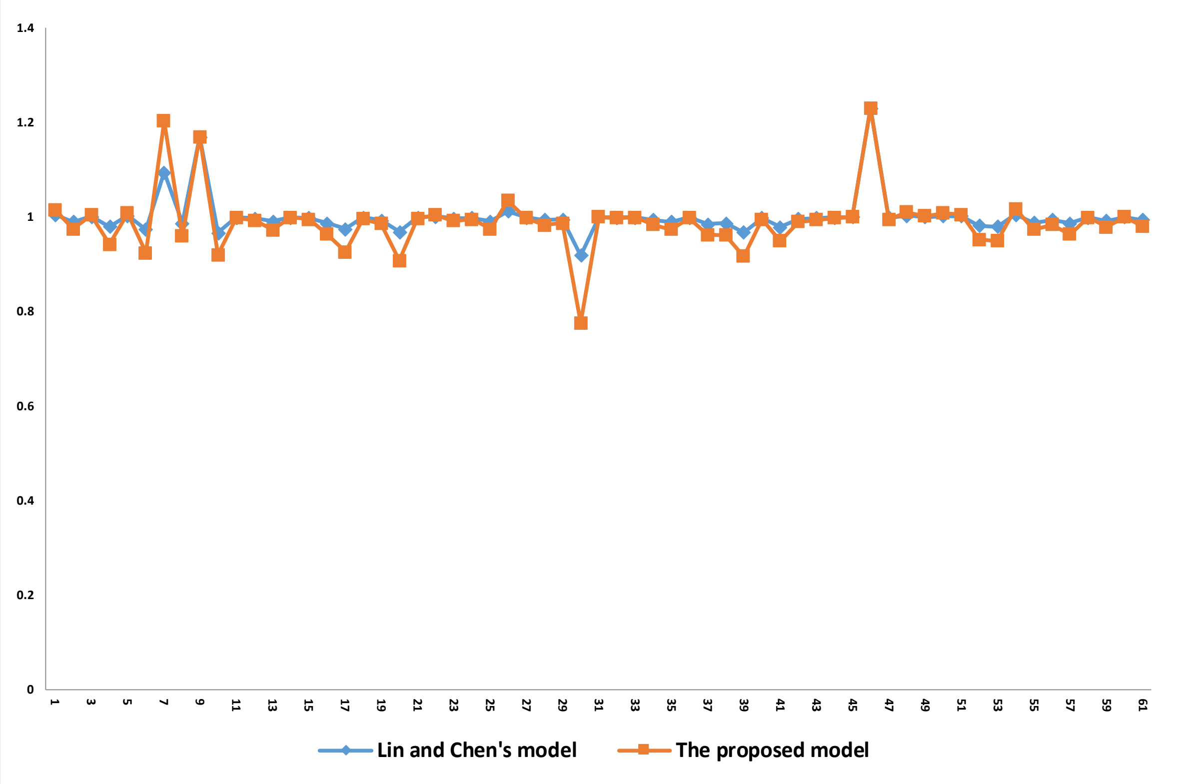

Figure 2. Comparison of efficiency score between Lin and Chen's model and the proposed model.

Table 12 shows the target input-output values of inefficient units determined by Lin and Chen's model and the proposed model.

Table 12. Improved targets for inefficient units provided by Lin and Chen's model and the proposed model

Banks	Input targets (million $)						Output targets (million $)			
	Proposed model			Lin & Chen's model			Proposed model		Lin & Chen's model	
	ASST	EQTY	DEPO	ASST	EQTY	DEPO	LOAN	PROF	LOAN	PROF
2	6833.854	1159.259	5281.466	6617.272	1117.092	5119.522	4887.497	167.0831	4782.684	162.2876
4	2699.356	443.5269	2280.62	2496.008	408.551	2109.128	2065.827	71.9906	1980.148	66.5723
6	2984.643	692.665	1912.941	2747.486	626.0042	1777.93	2350.448	69.2342	2220.816	62.2433
8	409.1027	257.7952	182.0847	403.3107	245.406	186.1356	398.9508	10.4053	391.2029	8.6938
10	9558.56	761.3459	8248.126	8565.898	682.8822	7390.993	4815.444	179.6554	4412.044	159.835
11	507.1023	97.82	405.1641	506.7941	97.6627	404.9726	438.8196	5.2691	438.4429	5.2024
12	374.109	57.3863	314.2385	373.3876	57.0744	313.6925	296.8849	-13.414	296.0455	-13.5934
13	730.147	138.5945	614.449	718.7452	134.7874	605.155	646.8699	19.6006	636.9193	18.2648
14	876.3263	121.8419	723.199	875.712	121.7485	722.697	724.9398	21.1566	724.6789	21.1254
15	3873.926	388.0565	3231.567	3859.332	386.3531	3219.74	3204.826	60.1916	3193.218	59.7731
16	2182.509	266.9593	1709.104	2071.636	253.3934	1622.288	1480.401	55.3687	1444.502	52.5598
17	2587.745	344.9764	2300.374	2437.568	318.8595	2163.655	2200.653	45.1771	2080.813	39.635

18	510.4718	106.9281	403.0567	508.2422	106.4609	401.2964	362.2116	15.8594	361.6827	15.7128
19	11530.17	1242.47	9656.851	11299.11	1214.765	9465.86	5745.236	273.4285	5651.285	268.009
20	3167.718	405.8167	2546.688	2933.961	367.4686	2369.061	2680.296	53.3772	2489.353	45.59
21	724.5993	102.48	616.4587	722.5455	102.1793	614.7132	573.6631	17.4778	572.9657	17.3723
23	5947.753	689.233	4010.617	5892.144	682.2353	3976.109	3252.709	131.6787	3232.844	130.52
24	1562.795	206.3604	1206.272	1552.081	205.0158	1197.878	869.5391	42.0061	867.3114	41.7293
25	1325.591	168.6568	1159.627	1280.482	162.9441	1120.174	903.6532	34.7996	890.1801	33.3227
27	7361.692	1224.742	5093.056	7347.426	1221.764	5084.39	5618.155	152.6258	5609.497	152.3051
28	881.4456	144.6448	733.8897	872.186	142.2805	726.4742	768.0558	22.6014	760.6852	21.763
29	6306.268	828.2501	4872.077	6209.315	813.9194	4800.005	3594.536	155.1399	3556.642	152.8911
30	11117.49	637.755	5207.498	7838.055	461.3808	3864.926	4482.981	138.6569	3508.525	96.2883
32	714.0055	151.19	555.6718	712.0383	150.8746	554.0765	533.274	23.3927	532.9347	23.3315
33	8296.48	851.4471	6418.351	8290.827	850.7763	6414.35	6034.671	178.013	6030.717	177.8599
34	1571.465	316.8273	1003.714	1551.683	311.0659	994.3586	1364.418	37.4559	1350.229	36.4466
35	5705.575	872.8085	4589.198	5517.134	840.698	4440.241	3533.535	143.6078	3458.708	139.3407
36	511.6953	206.6401	298.631	509.4482	206.1848	296.9301	150.6965	20.5879	150.6726	20.5304
37	9544.164	971.5707	8340.758	9154.37	926.3216	7996.735	7146.104	190.7355	6858.962	180.8549
38	1826.062	323.7825	1493.601	1725.153	306.2472	1410.903	1273.211	52.348	1241.377	49.4927
39	7192.09	1388.119	5922.661	6401.541	1213.949	5278.975	2805.214	178.8193	2593.98	160.7339
40	603.3752	158.7216	438.8434	596.649	157.5208	433.532	438.5947	22.0052	437.66	21.7903
41	16586.16	1732.263	14142.14	15434.78	1600.563	13164.75	7765.635	389.6942	7297.072	362.6828
42	2108.36	226.864	1763.168	2094.683	224.9967	1752.108	1793.545	-0.153	1781.627	-0.5153
43	644.3943	145.5972	497.4311	638.8151	144.4783	493.0367	538.9084	20.8951	537.4154	20.6382
44	616.253	178.8509	398.9968	615.9367	178.6042	398.9506	582.7374	15.0955	582.3349	15.0379
47	806.7178	110.7147	689.9633	801.8951	110.0368	685.8403	630.3845	20.028	628.7614	19.8017
52	7170.236	1301.942	5845.49	6821.406	1217.42	5570.859	5832.108	172.0372	5576.113	162.0398
53	16989.77	2227.197	13957.01	15814.53	2050.354	13009.8	6892.642	403.5169	6493.494	376.7259
55	11593.55	1088.959	9942.374	11174.81	1047.08	9584.043	5670.454	269.1724	5497.787	259.2105
56	6726.225	578.509	5809.205	6603.553	567.7041	5703.139	3296.105	149.8845	3250.942	147.0272
57	4263.575	496.6127	3616.459	4083.508	473.6765	3464.458	2473.862	105.8998	2403.911	101.2353
58	1024.241	252.2266	27.9691	1023.128	251.9474	27.9636	347.5298	13.9444	347.4031	13.9081
59	3405.714	416.7961	2979.082	3322.783	405.7885	2906.263	2417.741	84.5865	2381.133	82.4255
61	402.9495	169.8205	27.4778	393.6935	165.6988	27.3614	120.97	10.6846	120.97	9.9117

It could be concluded from Table 12, both models provide the improved targets for all inefficient DMUs. Under the proposed model, the inputs and the outputs of each inefficient unit should be reduced and expanded, respectively, in order to reach the super-efficiency frontier. Hence, the proposed model the same as Lin and Chen's model can provide improved target inputs and outputs for all the inefficient units but there are variations due to having different directions in their movements towards the super-efficiency frontier. From the theoretical analysis and the above examples, it is concluded that the proposed model can deal with the data set with free in sign values and can provide improved targets for inefficient units. In addition, the proposed model takes desirable properties of monotonicity, unit and translation invariance. The model fully eliminates the infeasibility issue of the VE model and successfully addresses the shortcomings of Lin and Chen's model. More importantly, different from current DEA models handling negative data, the proposed model can provide a complete ranking order for all the DMUs via a new super-efficiency measure.

6. Conclusion

Conventional DEA models are introduced to evaluate units with non-negative data, while in practice there are important units with negative data and they need to be evaluated. Super-efficiency model in the presence of negative data is a relatively neglected issue in the DEA field. The existing super-efficiency models have some shortcomings in practice. In this study, by using a new non-negative direction, a novel radial DDF-based super-efficiency model is proposed to make a distinction between efficient and inefficient units. The model can provide a complete ranking order for all the DMUs via a new super-efficiency measure. The model guaranties the feasibility no matter whether the input-outputs data are non-negative or not. It addresses the infeasibility issues occur in NLS model and contains advantages such as monotonicity, unit invariance and translation invariance properties. Apart from numerical examples, an empirical study in bank sector demonstrates the reliability and superiority of the proposed model in distinguishing units.

References

Ali, A.I., and Seiford, L.M. (1990). "Translation invariance in data envelopment analysis", Operations Research Letters, Vol. 9, pp. 403-405.

Andersen, P., and Petersen, N.C., (1993). "A procedure for ranking efficient units in data envelopment analysis", Management Science, Vol. 39, No. 10, pp. 1261–1294.

Banker, R.D., Charnes A., and Cooper W.W., (1984). "Some models for estimating technical and scale inefficiencies in data envelopment analysis", Management Science, Vol. 30, No. 9, pp. 1078-1092.

Basso, A., and Funari, S., (2014). "Constant and variable returns to scale DEA models for socially responsible investment funds", European Journal of Operational Research, 235(3), 775-783.

Briec, W., (1997). "Agraph-type extension of Farrell technical efficiency measure", Journal of Productivity Analysis, Vol. 8, No. 1, pp. 95-110.

Briec, W., and Kerstens, K., (2009). "Infeasibilities and directional distance functions with application to the determinateness of the Luenberger productivity indicator", Journal of Optimization Theory and Applications, Vol. 141, No. 1, pp. 55-73.

Chambers, R., Chung, Y., and Färe, R., (1998). "Profit, directional distance functions, and Nerlovian efficiency", Journal of Optimization Theory and Applications, Vol. 98, pp. 351–364.

Chambers, R. G., Chung, Y., and Färe, R., (1996). “Benefit and distance functions”, Journal of Economic Theory, Vol. 70, pp. 407–419.

Charnes A., Cooper W. W., and Rhodes E., (1978). “Measuring the efficiency of decision making units”, European Journal of Operational Research, Vol. 2, No. 6, pp. 429-444.

Chen, Y., (2005). “Measuring super-efficiency in DEA in the presence of infeasibility”, European Journal of Operational Research, Vol. 61, pp. 545–551.

Chen, Y., Du, J., and Hoa, J., (2013). “Super-efficiency based on a modified directional distance function”, Omega, Vol. 41, pp. 621-625.

Chen, Y., and Liang, L. (2011). “Super-efficiency DEA in the presence of infeasibility: One model approach”, European Journal of Operational Research, Vol. 213, pp. 359–360.

Cheng, G., Zervopoulos, P., and Qian, Z., (2013). “A variant of radial measure capable of dealing with negative inputs and outputs in data envelopment analysis”, European Journal of Operational Research, Vol. 225, No. 1, pp. 100–105.

Cook, W.D., Liang, L., Zha, Y., and Zhu, J., (2009). “A modified super-efficiency DEA model for infeasibility”, Journal of Operational Research Society, Vol. 60, pp. 276–281.

Emrouznejad, A., Anouze, A.L., and Thanassoulis, E., (2010). “A semi-oriented radial measure for measuring the efficiency of decision making units with negative data, using DEA”, European Journal of Operational Research, Vol. 200, No. 1, pp. 297-304.

Emrouznejad, A. and Anouze, A.L., (2010). “Data envelopment analysis with classification and regression tree - A case of banking efficiency”, Expert Systems with Applications, Vol. 27, No. 4, pp. 231-246.

Hadi-Vencheh, A., and Esmaeilzadeh, A., (2013). “A new super-efficiency model in the presence of negative data”, Journal of the Operational Research Society, Vol. 64, No. 3, pp. 396-401.

Kerstens, K., and Van de Woestyne, I., (2011). “Negative data in DEA: a simple proportional distance function approach”, Journal of the Operational Research Society, Vol. 62, No. 7, pp. 1413-1419.

Kerstens, K., and VandeWoestyne, I., (2014). “A note on a variant of radial measure capable of dealing with negative inputs and outputs in DEA”, European Journal of Operational Research, Vol. 234, No. 1, pp. 341-342.

Lee, H. S., and Zhu, J., (2012). “Super-efficiency infeasibility and zero data in DEA”, European Journal of Operational Research, Vol. 216, pp. 429–433.

Lee, H. S., Chu, C. W., and Zhu, J., (2011). “Super-efficiency DEA in the presence of infeasibility”, European Journal of Operational Research, Vol. 212, pp. 141–147.

Lin, R., and Chen, Z., (2015). “Super-efficiency measurement under variable return to scale: an approach based on a new directional distance function”, Journal of the Operational Research Society Vol. 66, pp. 1506–1510.

Lin, R., and Chen, Z., (2017). “A directional distance based super-efficiency DEA model handling negative data”, Journal of the Operational Research Society, Vol. 68, No. 11, pp. 1312-1322.

Lovell, C.A.K., (1995). “Measuring the Macroeconomic Performance of the Taiwanese Economy”, International Journal of Production Economics, Vol. 39, pp. 165-178.

Lovell, C.A.K., and Pastor, J.T., (1995), “Units invariant and translation invariant DEA models”, Operations Research Letters, 18(3), 147-151.

Pastor, J.T., and Aparicio, J., (2015). “Translation Invariance in Data Envelopment Analysis”, In Data Envelopment Analysis, pp. 245-268.

Pastor, J.T., and Ruiz, J.L., (2007). “Variables with negative values in DEA. In Modeling data irregularities and structural complexities in data envelopment analysis, pp. 63-84.

Pastor, J.T., (1996). “Translation invariance in data envelopment analysis: A generalization”, Annals of Operations Research, Vol. 66, pp. 93-102.

Pourmahmoud, J., Hatami-Marbini, A., and Babazadeh, E., (2016). “A comment on a new super-efficiency model in the presence of negative data”, Journal of the Operational Research Society, Vol. 67, No. 3, pp. 530-534.

Ray, S. C., (2008). “The directional distance function and measurement of super-efficiency: an application to airlines data”, Journal of the Operational Research Society, Vol. 59, No. 6, pp. 788-797.

Seiford, L.M., and Zhu, J. (1999). Infeasibility of super-efficiency data envelopment analysis models, pp. 174–187.

Silva Portela, M.C.A., Thanassoulis, E., and Simpson, G., (2004). “A directional distance approach to deal with negative data in DEA: An application to bank branches”, Journal of Operational Research Society, Vol. 55, No. 10, pp. 1111-1121.

Sharp, J. A., Meng, W., and Liu, W. (2007). “A modified slacks-based measure model for data envelopment analysis with ‘natural’ negative outputs and inputs”, Journal of the Operational Research Society, Vol. 58, No. 12, pp. 1672-1677.

Appendix A

Theorem 1. Model (5) is always feasible and the following inequalities are hold:

a) $0 < \rho^* \le 1$ for $(x_{io}, y_{ro}) \in T_o^s$;
b) $1 < \rho^* \le 2$ for $(x_{io}, y_{ro}) \notin T_o^s$.

Proof. let $J'_o = \{i | x_i^{max} - x_{io} > 0, \mathrm{i} = 1,2,\dots,\mathrm{m}\}$ and $O'_o = \{i | x_i^{max} - x_{io} = 0, \mathrm{i} = 1,2,\dots,\mathrm{m}\}$ for each $o \in \{1,2,\dots,n\}$. Thus, $x_i^{max} - x_{io} \ge 0$ implies that $J'_o \cup O'_o = \{\mathrm{r} = 1,2,\dots,\mathrm{s}\}$. Due to convexity constraint i.e. $\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j = 1$, we have

$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij} \le \max_{j\neq o}\{x_{ij}\} \le \max_{j}\{x_{ij}\} = x_i^{max} = x_{io}, \ \mathrm{i} \in O'_o.$

This shows that the input constraints in model (5) satisfy for all $\mathrm{i} \in O'_o$. Hence, the input constraints in model (5) are equivalent to

$$\delta \le \frac{x_{io} - \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij}}{x_i^{max} - x_{io}}, \ \mathrm{i} \in J'_o. \tag{7}$$

Correspondingly, let $J_o = \{r | y_{ro} - y_r^{min} > 0, \mathrm{r} = 1,2,\dots,\mathrm{s}\}$ and $O_o = \{r | y_{ro} - y_r^{min} = 0, \mathrm{r} = 1,2,\dots,\mathrm{s}\}$ for each $o \in \{1,2,\dots,n\}$. Thus, $y_{ro} - y_r^{min} \ge 0$ implies that $J_o \cup O_o = \{\mathrm{r} = 1,2,\dots,\mathrm{s}\}$. Due to convexity constraint, we have

$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} \ge \min_{j\neq o}\{y_{rj}\} \ge \min_{j}\{y_{rj}\} = y_r^{min} = y_{ro}, \ \mathrm{r} \in O_o.$

This shows that the output constraints in model (5) satisfy for all $\mathrm{r} \in O_o$. Hence, the output constraints in model (5) are equivalent to

$$\delta \le \frac{\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} - y_{ro}}{y_{ro} - y_r^{min}}, \ \mathrm{r} \in J_o. \tag{8}$$

There are two cases as follows:

Case (I) when $(x_{io}, y_{ro}) \in T_o^s$:

We have $\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij} \le x_{io}$ and $\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} \ge y_{ro}$ for $i = 1,2,\dots,m$ and $r = 1,2,\dots,s$, respectively. So,

$$\frac{x_{io} - \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij}}{x_i^{max} - x_{io}} \ge 0, \quad \mathrm{i} \in J'_o, \tag{9}$$

$$\frac{\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} - y_{ro}}{y_{ro} - y_r^{min}} \ge 0, \ \mathrm{r} \in J_o. \tag{10}$$

Inequalities of (7)-(10) result that $\delta = 0$ is a feasible solution of model (5), and consequently $\delta^* \ge 0$ always hold for $o \in \{1,2,\dots,n\}$.

Case (II) when $(x_{io}, y_{ro}) \notin T_o^s$:

In this case $\exists i$: $\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij} > x_{io}$ and/or $\exists r$: $\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} < y_{ro}$ which implies that $x_{io} - \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij} < 0$ and/or $\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} - y_{ro} < 0$. Due to (7) and (8), model (5) is still feasible and $\delta^* < 0$ is the optimal solution.

In addition, according to model (5)

$$x_i^{min} \le x_{io} - (x_i^{max} - x_{io})\delta \le x_i^{max}$$
$$y_r^{min} \le y_{ro} + (y_{ro} - y_r^{min})\delta \le y_r^{max}$$
$$\frac{x_{io} - x_i^{max}}{(x_i^{max} - x_{io})} \le \delta \le \frac{x_{io} - x_i^{min}}{(x_i^{max} - x_{io})}$$
$$\frac{y_r^{min} - y_{ro}}{(y_{ro} - y_r^{min})} \le \delta \le \frac{y_r^{max} - y_{ro}}{(y_{ro} - y_r^{min})}$$

Thus,

$$-1 \le \delta \le \frac{x_{io} - x_i^{min}}{(x_i^{max} - x_{io})}$$
$$-1 \le \delta \le \frac{y_r^{max} - y_{ro}}{(y_{ro} - y_r^{min})}$$

Therefore,

$$-1 \le \delta^* \le \hat{\delta}_o. \tag{11}$$

It is evident that $\hat{\delta}_o > 0$ for all units whether $(x_{io}, y_{ro}) \in T_o^s$ or $(x_{io}, y_{ro}) \notin T_o^s$.
When $(x_{io}, y_{ro}) \in T_o^s$, from (11) we have $\delta^* \geq 0$. Thus, $0 \leq \delta^* < \hat{\delta}_o$. Therefore, $0 \leq \frac{\delta^*}{\hat{\delta}_o} < 1$. Consequently, $0 < \rho^* \leq 1$ for inefficient units.

Moreover, when $(x_{io}, y_{ro}) \notin T_o^s$ from (11) we have $-1 \leq \delta^* < 0$ which implies that $-1 \leq \frac{-1}{\hat{\delta}_o} \leq \frac{\delta^*}{\hat{\delta}_o} < 0$. Thus, $1 < \rho^* \leq 2$ for efficient units. ■

Appendix B

Theorem 2. Model (5) is unit and translation invariant.

Proof.
(i)To show the units invariance of model (5), assume that the inputs $\mathrm{x_{ij}}$ and outputs $\mathrm{y_{rj}}$ are multiplied by the positive α_i and μ_r, respectively. Let $\tilde{x}_{ij} = \alpha_i x_{ij}$ $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$, $\tilde{y}_{rj} = \mu_r y_{rj}$ $(r = 1, 2, \dots, s;\ j = 1, 2, \dots, n)$, $\tilde{x}_i^{max} = \max_{j=1,2,\dots,n}\{\tilde{x}_{ij}\}$ $(i = 1, 2, \dots, m)$ and $\tilde{y}_r^{min} = \min_{j=1,2,\dots,n}\{\tilde{y}_{rj}\}$ $(r = 1, 2, \dots, s)$.
Hence, the model (5) using the transformed date is written as following:

$$\begin{aligned} &\max \quad \delta \\ &s.t.\quad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \tilde{x}_{ij} \leq \tilde{x}_{io} - (\tilde{x}_i^{max} - \tilde{x}_{io})\delta, \quad \forall i, \\ &\qquad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \tilde{y}_{rj} \geq \tilde{y}_{ro} + (\tilde{y}_{ro} - \tilde{y}_r^{min})\delta, \quad \forall r, \\ &\qquad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j = 1, \\ &\qquad \lambda_j \geq 0, \ \ \forall j,\ j \neq o \end{aligned}$$

This model is transformed to the model (5), in terms of the untransformed data, after substitution of $\alpha_i x_{ij}$ for $\tilde{x}_{ij}$ in the input constraints and $\mu_r y_{rj}$ for $\tilde{y}_{rj}$ in the output constraints, and cancellation of the common factors from both sides of the inequalities.

(ii)To show the translation invariance of model (5), assume that the inputs $\mathrm{x_{ij}}$ and outputs $\mathrm{y_{rj}}$ are transformed by the γ_i and σ_r, respectively. Let $\tilde{x}_{ij} = \gamma_i + x_{ij}$ $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$, $\tilde{y}_{rj} = \sigma_r + y_{rj}$ $(r = 1, 2, \dots, s;\ j = 1, 2, \dots, n)$, $\tilde{x}_i^{max} = \max_{j=1,2,\dots,n}\{\tilde{x}_{ij}\}$ $(i = 1, 2, \dots, m)$ and $\tilde{y}_r^{min} = \min_{j=1,2,\dots,n}\{\tilde{y}_{rj}\}$ $(r = 1, 2, \dots, s)$.
Hence, model (5) in terms of the transformed date is written as following:

$$\begin{aligned} &\max \quad \delta \\ &s.t.\quad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \tilde{x}_{ij} \leq \tilde{x}_{io} - (\tilde{x}_i^{max} - \tilde{x}_{io})\delta, \quad \forall i, \\ &\qquad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j \tilde{y}_{rj} \geq \tilde{y}_{ro} + (\tilde{y}_{ro} - \tilde{y}_r^{min})\delta, \quad \forall r, \\ &\qquad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j = 1, \\ &\qquad \lambda_j \geq 0, \ \ \forall j,\ j \neq o \end{aligned}$$

The model is equivalent with the following problem:

$$\begin{aligned} &\max \quad \delta \\ &s.t.\quad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j x_{ij} + \left(\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j\right)\gamma_i \leq x_{io} + \gamma_i - (x_i^{max} - x_{io})\delta, \quad \forall i, \\ &\qquad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j y_{rj} + \left(\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j\right)\sigma_r \geq y_{ro} + \sigma_r + (y_{ro} - y_r^{min})\delta, \quad \forall r, \\ &\qquad \sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j = 1, \\ &\qquad \lambda_j \geq 0, \ \ \forall j,\ j \neq o \end{aligned}$$

Due to the convexity condition, this model is transformed to the model (5) in terms of the untransformed data, after cancellation of the common factors from both sides of the inequalities.

Appendix C

To show the monotonicity property of Model (5) suppose that the inputs and the outputs of DMU_o are reduced by $\Delta \mathrm{x_{io}}$ and increased by Δy_{ro}, respectively; and let $\mathrm{x_{io}} \geq 0$, i = 1, 2, ..., m, and $y_{\mathrm{ro}} \geq 0$, r = 1, 2, ..., s. Note that here $\Delta \mathrm{x_{io}} \geq 0$, i = 1, 2, ..., m and $\Delta y_{\mathrm{ro}} \geq 0$, r = 1, 2, ..., s. Since the input and output data of DMU_o are changed, the constants x_i^{max} and y_r^{min} should be adjusted correspondingly. However, due to the non-negativity of $\Delta \mathrm{x_{io}}$ and Δy_{ro}, the definition of x_i^{max} and y_r^{min} is not changed:

$$x_i^{max} = max\{x_{ij}, \forall j, \mathrm{x_{io}} - \Delta \mathrm{x_{io}}\} = \max_{j=1,2,\ldots,n}\{x_{ij}\},\ i = 1,2,\ldots,m$$

and

$$y_r^{min} = min\{y_{rj}, \forall j, y_{\mathrm{ro}} + \Delta y_{\mathrm{ro}}\} = \min_{j=1,2,\ldots,n}\{y_{rj}\},\ r = 1,2,\ldots,s.$$

Therefore, the following conclusion is made.

Theorem 4. If inputs (outputs) of the DMU_o are reduced (increased), the optimal value of model (5) does not increase.

Proof. If specified input reduction and output expansion happens, the direction vector is

$(x_i^{max} - (\mathrm{x_{io}} - \Delta \mathrm{x_{io}}), (\mathrm{y_{ro}} + \Delta \mathrm{y_{ro}}) - y_r^{min})$ and the following statement is made:

$x_i^{max} - (\mathrm{x_{io}} - \Delta \mathrm{x_{io}}) \geq 0$, i = 1, 2, . . . ,m, and $\mathrm{y_{ro}} + \Delta \mathrm{y_{ro}} - y_r^{min} \geq 0$, r = 1, 2, . . . , s.

Consequently the corresponding model (5) for the DMU_o is rewritten as

$$\begin{aligned} &\max \quad \delta \\ &s.t. \quad \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j x_{ij} \leq (\mathrm{x_{io}} - \Delta \mathrm{x_{io}}) - (x_i^{max} - (\mathrm{x_{io}} - \Delta \mathrm{x_{io}}))\delta, \quad \forall i, \\ &\quad \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j y_{rj} \geq (\mathrm{y_{ro}} + \Delta \mathrm{y_{ro}}) + ((\mathrm{y_{ro}} + \Delta \mathrm{y_{ro}}) - y_r^{min})\delta, \quad \forall r, \\ &\quad \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j = 1, \\ &\quad \lambda_j \geq 0, \ \forall j,\ j \neq o \end{aligned} \tag{12}$$

Assume the optimal solution of model (12) as (λ_j', δ'). A similar derivation as that for (9) and (10), the input and output constraints of model (12) are equivalent to the following, respectively:

$$\delta' \leq \frac{\mathrm{x_{io}} - \Delta \mathrm{x_{io}} - \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j' \mathrm{x_{ij}}}{x_i^{max} - (\mathrm{x_{io}} - \Delta \mathrm{x_{io}})}, \quad \mathrm{i} \in J_o', \tag{13}$$

$$\delta' \leq \frac{\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j' \mathrm{y_{rj}} - (\mathrm{y_{ro}} + \Delta \mathrm{y_{ro}})}{y_{ro} + \Delta \mathrm{y_{ro}} - y_r^{min}}, \ \mathrm{r} \in J_o \tag{14}$$

Where $J_o' = \{i | x_i^{max} - (\mathrm{x_{io}} - \Delta \mathrm{x_{io}}) > 0, \mathrm{i} = 1,2,\ldots,\mathrm{m}\}$ and $J_o = \{r | y_{ro} + \Delta \mathrm{y_{ro}} - y_r^{min} > 0, \mathrm{r} = 1,2,\ldots,\mathrm{m}\}$. Due to deviation, the following statements are hold:

$$x_i^{min} = min\{x_{ij}, \forall j, \mathrm{x_{io}} - \Delta \mathrm{x_{io}}\},\ i = 1,2,\ldots,m$$

and

$$y_r^{max} = max\{y_{rj}, \forall j, y_{\mathrm{ro}} + \Delta y_{\mathrm{ro}}\},\ r = 1,2,\ldots,s.$$

Obviously,

$$\frac{\mathrm{x_{io}} - \Delta \mathrm{x_{io}} - \sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j' \mathrm{x_{ij}}}{x_i^{max} - (\mathrm{x_{io}} - \Delta \mathrm{x_{io}})} \geq \frac{\mathrm{x_{io}} - \Delta \mathrm{x_{io}} - x_i^{max}}{x_i^{max} - (\mathrm{x_{io}} - \Delta \mathrm{x_{io}})} \geq -1, \quad \mathrm{i} \in J_o', \tag{15}$$

$$\frac{\sum_{\substack{j=1 \\ j\neq o}}^{n} \lambda_j' \mathrm{y_{rj}} - (\mathrm{y_{ro}} + \Delta \mathrm{y_{ro}})}{y_{ro} + \Delta \mathrm{y_{ro}} - y_r^{min}} \geq \frac{y_r^{min} - (\mathrm{y_{ro}} + \Delta \mathrm{y_{ro}})}{y_{ro} + \Delta \mathrm{y_{ro}} - y_r^{min}} \geq -1,\ \mathrm{r} \in J_o. \tag{16}$$

Since we maximize δ in model (12), $\delta' \geq -1$ always hold for $o \in \{1,2,\dots,n\}$ due to (13), (14), (15) and (16). Then, $1+\delta'$ is non-negative. In this regards,

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j' x_{ij} \leq (x_{io} - \Delta x_{io})(1+\delta') - x_i^{max}\delta' \leq x_{io}(1+\delta') - x_i^{max}\delta' = x_{io} - (x_i^{max} - x_{io})\delta', \quad \forall i, \tag{17}$$

$$\sum_{\substack{j=1\\ j\neq o}}^{n} \lambda_j' y_{rj} \geq (y_{ro} + \Delta y_{ro})(1+\delta') - y_r^{min}\delta' \geq y_{ro}(1+\delta') - y_r^{min}\delta' = y_{ro} + (y_{ro} - y_r^{min})\delta', \quad \forall r \tag{18}$$

Therefore, (λ_j', δ') is a feasible solution for model (12). Maximizing of δ' is aimed in model (5), hence $\delta^* \geq \delta'$.